Dynamical system analysis of K-essence

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1 Introduction

In Quintessence, the late-time acceleration is driven by the potential energy of the scalar field. If the potential dominates the kinetic energy the effective pressure becomes negative giving rise to acceleration. However, it is possible to have accelerated expansion with kinetic energy only. Originally this idea was introduced to describe inflation of the early universe and this model was named as K-inflation (the model was motivated from string theoretic calculations) [1]. Then this concept was used in explaining the late time acceleration [2]. In K-essence the Lagrangian contains a non-canonical kinetic energy term. A generic action in K-essence is given by,

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}(\phi, X) \tag{1}$$

where X is given by $X = \frac{1}{2}\dot{\phi}^2$. We take $\mathcal{L}$ as a polynomial of X of degree 2, where the coefficients are functions of ϕ,

$$\mathcal{L} = \alpha(\phi)X + \beta(\phi)X^2 - V(\phi) \tag{2}$$

We take a new approach to this model, where we analyse the model using stability analysis of a dynamical system. The purpose of this analysis is twofold:

- To see whether V(ϕ) = 0 can give rise to acceleration. If it can, then whether there are any constraints on $\alpha(\phi)$ and $\beta(\phi)$.
- To analyse the stability of the model with $V(\phi) \neq 0$.

2 Friedmann Equations

For the given Lagrangian, the pressure and energy density is given by,

$$p_\phi = \mathcal{L}(\phi, X) = \alpha(\phi)X + \beta(\phi)X^2 - V(\phi) \tag{3a}$$

$$\rho_\phi = -\mathcal{L} + 2X\mathcal{L}_X = \alpha(\phi)X + 3\beta(\phi)X^2 + V(\phi) \tag{3b}$$

The Friedmann equations are given by

$$H^2 = \frac{1}{3}(\rho_b + \rho_\phi) \tag{4a}$$

$$\dot{H} = -\frac{1}{2}(\rho_m + \rho_\phi + p_m + p_\phi) \tag{4b}$$

We take dust as the baryogenic matter ($p_m = 0$) and substitute 4(a) in 4(b), and get,

$$\dot{H} = -\frac{1}{2}(p_\phi + 3H^2) \tag{5}$$

Thus we eliminate a variable ρ_m using the constraint equation 4(a). The corresponding Klein-Gordon equation (or the conservation equation) is given by,

$$\ddot{\phi}\big[\alpha(\phi) + 3\beta(\phi)\dot{\phi}^2\big] + \alpha'(\phi)\frac{\dot{\phi}^2}{2} + 3\beta'(\phi)\frac{\dot{\phi}^4}{4} + 3H\dot{\phi}\big[\alpha(\phi) + \beta(\phi)\dot{\phi}^2\big] = 0 \tag{6}$$

3 Dynamic System Analysis

The system has five unknowns viz. a(t), ϕ, $\alpha(\phi)$, $\beta(\phi)$, and $V(\phi)$. To describe the system we define five variables,

$$x = \frac{\sqrt{\alpha(\phi)}\dot{\phi}}{\sqrt{6}H} \tag{7a}$$

$$y = \frac{\sqrt{\beta(\phi)}\dot{\phi}^2}{2\sqrt{3}H} \tag{7b}$$

$$z = \sqrt{\frac{V}{3H^2}} \tag{7c}$$

$$\lambda = \frac{1}{\alpha}\frac{d\alpha}{d\phi}\frac{\dot{\phi}}{H} \tag{7d}$$

$$\delta = \frac{1}{\beta}\frac{d\beta}{d\phi}\frac{\dot{\phi}}{H} \tag{7e}$$

The corresponding differential equations can be written as,

$$x' = \frac{3}{2}x(x^2+y^2-z^2+1)+\frac{\lambda}{2}x - x\left[\frac{3(x^2+2y^2)}{x^2+6y^2}+\frac{\lambda x^2+3\delta y^2+\sigma z^2}{2(x^2+6y^2)}\right] \tag{8a}$$

$$y' = \frac{3}{2}y(x^2+y^2-z^2+1)+\frac{\delta}{2}y - 2y\left[\frac{3(x^2+2y^2)}{x^2+6y^2}+\frac{\lambda x^2+3\delta y^2+\sigma z^2}{2(x^2+6y^2)}\right] \tag{8b}$$

$$z' = \frac{3}{2}z(x^2+y^2-z^2+1)+\frac{\sigma}{2}z \tag{8c}$$

$$\lambda' = \frac{3}{2}\lambda(x^2+y^2-z^2+1)-\lambda^2+\lambda^2\Gamma - \lambda\left[\frac{3(x^2+2y^2)}{x^2+6y^2}+\frac{\lambda x^2+3\delta y^2+\sigma z^2}{2(x^2+6y^2)}\right] \tag{8d}$$

$$\delta' = \frac{3}{2}\delta(x^2+y^2-z^2+1)-\delta^2+\delta^2\tau - \lambda\left[\frac{3(x^2+2y^2)}{x^2+6y^2}+\frac{\lambda x^2+3\delta y^2+\sigma z^2}{2(x^2+6y^2)}\right] \tag{8e}$$

where $\Gamma = \frac{\alpha(\frac{d^2\alpha}{d\phi^2})}{(\frac{d\alpha}{d\phi})^2}$, $\tau = \frac{\beta(\frac{d^2\beta}{d\phi^2})}{(\frac{d\beta}{d\phi})^2}$, and $\sigma = \frac{d(\ln V)}{d\xi}$ are parameters. All the primes are with respect to ξ where ξ =ln(a).

Case I: V(ϕ) = 0, λ and δ are constants.

For this case, the RHS of eqn. 8(c)-(e) are zero along with z=0. The corresponding fixed points for the system described by 8(a)-(b) are given in Table 1.

Point	x	y	Eigenvalues
A.	0	$\pm\frac{1}{\sqrt{3}}$	$\{(1-\frac{\delta}{4}+\frac{\lambda}{2}),\ 1\}$
B.	± 1	0	$\{3,\ (-3+\frac{\delta}{2}-\lambda)\}$
C.	$\pm\sqrt{\frac{\delta-2\lambda-4}{2}}$	$\pm\sqrt{\frac{6-\delta+2\lambda}{6}}$	$\{\frac{(2\lambda-\delta+6)(2\lambda-\delta+4)}{-2\lambda+\delta-8},\ (\delta-2\lambda-3)\}$

Clearly the fixed points A and B are either unstable node or saddle point depending upon the values λ and δ. However the final fixed point can give rise to a stable node. The condition for point C to be a stable fixed point is,

$$2\lambda > \delta - 3 \tag{9}$$

Now, if the system has to have a stable fixed point, then the condition on the quantities λ and δ are given by (9). This is a constraint on the coefficients $\alpha(\phi)$ and $\beta(\phi)$. If we use this condition in the fixed pt. C we get imaginary values for x and real values for y. This imply that in order to have a stable fixed point, $\alpha(\phi)$ has to be negative and $\beta(\phi)$ has to be positive. This is another constraint on the coefficients of X.

$$\alpha(\phi) < 0, \qquad \beta(\phi) > 0 \tag{10}$$

Now in Table 2 we give the corresponding scalar field energy density parameter (Ω_ϕ) and effective equation of state γ_ϕ which are given by,

$$\Omega_\phi = x^2 + 3y^2 \tag{11a}$$

$$\gamma_\phi = \frac{2x^2 + 4y^2}{x^2 + 3y^2} \tag{11b}$$

Point	Ω_ϕ	γ_ϕ
A.	1	$\frac{4}{3}$
B.	1	1
C.	1	$\frac{\delta - 2\lambda}{3}$

So all of them result in a scalar field dominated universe but if a stable fixed point has to exist, then the final equation of state can be either a decelerated one or a accelerated one depending on the values of λ and δ. For accelerated universe $\gamma_\phi < \frac{2}{3}$. Now, for the universe going to a stable fixed point, if we impose the condition (9) we get, $\gamma_\phi < 1$. For an accelerated universe in the future we get,

$$2\lambda > \delta - 2 \tag{12}$$

Now we impose the current observations for scalar field energy parameter and effective equation of state,

$$\Omega_\phi = 0.685 \tag{13a}$$

$$\gamma_\phi = 0.05 \tag{13b}$$

Solving these two equations for x and y we get, x = 1.147i and y = 0.817. The imaginary value of x implies that the universe will evolve to point C if condition (9) is satisfied, otherwise it will diverge. It will be a accelerated expansion if condition (12) is satisfied. Otherwise, after a certain time the current acceleration will stop and deceleration will start.

Case II: V(ϕ) = 0, λ and δ are arbitrary.

In this case only z=0 and the system reduces to a 4-dimensional one. The corresponding fixed points are given in Table 3.

Point	x	y	λ	δ
A.	0	$\pm\frac{1}{\sqrt{3}}$	0	0
B.	0	$\pm\frac{1}{\sqrt{3}}$	0	$-\frac{4}{4\tau-5}$
C.	0	$\pm\frac{1}{\sqrt{3}}$	$\frac{1}{1-\Gamma}$	0
D.	0	$\pm\frac{1}{\sqrt{3}}$	$-\frac{4(\tau-1)}{(\Gamma-1)(4\tau-5)}$	$-\frac{4}{4\tau-5}$
E.	$\pm i\sqrt{2}$	± 1	0	0
F.	± 1	0	0	0

To find out the stability we analyse the eigenvalues for these fixed points which are given in Table 4,

Point	e_1	e_2	e_3	e_4	Stability
A.	1	1	1	1	unstable
B.	-1	1	1	$\frac{2\Gamma-3}{2(\Gamma-1)}$	unstable
C.	-1	1	$\frac{4(\tau-1)}{4\tau-5}$	$\frac{4(\tau-1)}{4\tau-5}$	unstable
D.	-1	1	$-\frac{4(\tau-1)}{4\tau-5}$	$\frac{2(2\Gamma-3)(\tau-1)}{(\Gamma-1)(4\tau-5)}$	unstable
E.	-3	-3	0	0	stable
F.	-3	3	0	0	unstable

So we see that only point E is a stable fixed point which has an imaginary value for x. This implies that $\alpha(\phi)$ has to be negative in order to obtain a stable fixed point for the system. The current observations given by (13) supports this fact and indicates that the universe will evolve to this fixed point if there is no potential.

The scalar energy density parameter and the effective equation of state for all the fixed points are listed in Table 5 below.

Point	Ω_ϕ	γ_ϕ
A, B, C, D	1	$\frac{4}{3}$
E.	1	0
F.	1	2

So all of them denotes a scalar field only universe scenario. Point E suggests the final equation of state will be 0 like the cosmological constant model. The other unstable fixed points indicate decelerated expansion phase. Hence we see that if we take the current observation as the initial condition the universe will continue its accelerated expansion phase until only dark energy contribution remains and the e.o.s. becomes that of the cosmological constant. It also indicates that it is possible to obtain the cosmological constant limit using K-essence only without any potential term. The only constraint that remains here is that $\alpha(\phi)$ has to be negative and $\beta(\phi)$ has to be positive.

Case III: V(ϕ) $\neq 0$, λ and δ are constants.

In this case we have z≠0. So we have the RHS of 8(d)-(e) as zero. The system reduces to a 3-dimensional system. The fixed points of this system are given in Table 6,

Point	x	y	z	Eigenvalues
A.	0	$\pm\frac{i\sqrt{\sigma}}{2\sqrt{3}}$	$\pm\frac{\sqrt{4+\sigma}}{2}$	$\{\frac{1}{4}(-\delta+2\lambda-\sigma),\ -(3+\sigma),\ -(4+\sigma)\}$
B	$\pm\frac{i\sqrt{\sigma}}{\sqrt{6}}$	0	$\pm\frac{\sqrt{6+\sigma}}{\sqrt{6}}$	$\{\frac{1}{2}(\delta-2\lambda+\sigma),\ -(3+\sigma),\ -(6+\sigma)\}$

So, for a stable fixed point to exist, the first condition is $\sigma > -3$. Now we list in Table 7 the scalar field energy density parameter and effective equation of state value for the above fixed points,

$$\Omega_\phi = x^2 + 3y^2 + z^2 \tag{14a}$$

$$\gamma_\phi = \frac{2x^2 + 4y^2}{x^2 + 3y^2 + z^2} \tag{14b}$$

Point	Ω_ϕ	γ_ϕ
A.	1	$-\frac{\sigma}{3}$
B.	1	$-\frac{\sigma}{3}$

So, if we omit the possibility of phantom fields, we must have the condition $\sigma < 0$. For $-3 < \sigma < 0$, both A and B can be stable fixed points depending upon λ and δ. For $2\lambda > \delta + \sigma$ point B is stable and for $2\lambda < \delta + \sigma$ point A is stable.

If we ignore phantom fields, we should note that $0 < \gamma_\phi < 1$, i.e. for $-2 < \sigma < 0$ we get the fate of the universe as continued accelerated expansion and for $-3 < \sigma < -2$ we get decelerated expansion as the future stage of the universe. So, there are both possibilities depending upon the slope of the potential.

Another thing to notice is that, for $y = 0$, the Lagrangian is basically the Lagrangian for Quintessence. Hence point B indicates a universe following the Quintessence scenario. Hence we can conclude that Quintessence can be obtained from K-essence as a stable solution of the system for $2\lambda > \delta + \sigma$.

In both Case I and Case III we note that all the fixed point corresponds to scalar field energy density parameter to be 1, i.e. in the future matter density will be zero. So, only scalar field will exist (i.e. only dark energy). Also we notice that if we allow phantom fields, then either $\alpha(\phi)$ has to be negative or $\beta(\phi)$ has to be negative.

Case IV: V(ϕ) $\neq 0$, λ and δ are arbitrary.

Now the system is 5-dimensional. The corresponding fixed points are given in Table 8,

Point	x	y	z	λ	δ
A.	0	$\pm\frac{i\sqrt{\sigma}}{2\sqrt{3}}$	$\pm\frac{\sqrt{4+\sigma}}{2}$	0	0
B.	0	$\pm\frac{i\sqrt{\sigma}}{2\sqrt{3}}$	$\pm\frac{\sqrt{4+\sigma}}{2}$	$\frac{\sigma}{4(\Gamma-1)}$	0
C.	0	$\pm\frac{i\sqrt{\sigma}}{2\sqrt{3}}$	$\pm\frac{\sqrt{4+\sigma}}{2}$	0	$\frac{\sigma}{4\tau-5}$
D.	0	$\pm\frac{i\sqrt{\sigma}}{2\sqrt{3}}$	$\pm\frac{\sqrt{4+\sigma}}{2}$	$\frac{\sigma(\tau-1)}{(4\tau-5)(\Gamma-1)}$	$\frac{\sigma}{4\tau-5}$
E.	$\pm\frac{i\sqrt{\sigma}}{\sqrt{6}}$	0	$\pm\frac{\sqrt{6+\sigma}}{\sqrt{6}}$	0	0

In this case also, if we avoid phantom fields, σ must be negative for the effective equation of state to be positive.

The corresponding eigenvalues for each point are given in Table 9 along with the condition for which the fixed points will be stable node,

Point	e_1	e_2	e_3	e_4	e_5	Stability Condition
A.	$-\frac{\sigma}{4}$	$-\frac{\sigma}{4}$	$-\frac{\sigma}{4}$	$-(\sigma+3)$	$-(\sigma+4)$	unstable
B.	$-\frac{\sigma}{4}$	$\frac{\sigma}{4}$	$-\frac{(2\Gamma-3)\sigma}{8(\Gamma-1)}$	$-(\sigma+3)$	$-(\sigma+4)$	unstable
C.	$\frac{\sigma}{4}$	$-\frac{(\tau-1)\sigma}{(4\tau-5)}$	$-\frac{(\tau-1)\sigma}{(4\tau-5)}$	$-(\sigma+3)$	$-(\sigma+4)$	$\sigma > -3$ $1 < \tau < 5/4$
D.	$\frac{\sigma}{4}$	$\frac{(\tau-1)\sigma}{(4\tau-5)}$	$-\frac{(2\Gamma-3)(\tau-1)\sigma}{2(\Gamma-1)(4\tau-5)}$	$-(\sigma+3)$	$-(\sigma+4)$	$\tau > 5/4$ or $\tau < 1$ $1 < \Gamma < 3/2$ $\sigma > -3$
E.	0	0	$\frac{\sigma}{2}$	$-(\sigma+3)$	$-(\sigma+4)$	$\sigma > -3$

We see that point A. and B. will never give rise to a stable fixed point. Depending upon very strict conditions on Γ and τ we will get stable fixed points for C and D. So, basically the choice for valid forms of $\alpha(\phi)$ and $\beta(\phi)$ are severely restricted. However point E gives a stable node without any further restriction on Γ and τ. Hence it allows more generic forms of $\alpha(\phi)$ and $\beta(\phi)$. Also we should keep in mind that this is the Quintessence case. Hence even if we start with K-essence, for point E we end up with Quintessence only and this is more generic choice since this allows more forms of $\alpha(\phi)$ and $\beta(\phi)$.

4 Summary

To summarize the discussion so far,

- **Case I : V(ϕ) = 0, λ and δ are constants.**
  - It is possible to obtain late time acceleration using K-essence without a potential term.
  - For the system to have a stable fixed point $2\lambda > \delta - 3$. This also implies that $\alpha(\phi) < 0$ and $\beta(\phi) > 0$. This constraint is also supported by the current observations of scalar energy density parameter and effective equation of state.
  - The final effective equation of state is always less than 1. For a continued accelerated expansion the condition is $2\lambda > \delta - 2$.
- **Case II : V(ϕ) = 0, λ and δ are variables.**
  - The only stable fixed point for the system is given by $x = \pm i\sqrt{2}$ and $y = \pm 1$. This implies that $\alpha(\phi) < 0$ and $\beta(\phi) > 0$ in absence of the potential.
  - The universe will reach the cosmological constant limit if there is no potential term in the K-essence. So, it is possible to reach cosmological constant limit without the potential in contrast to Quintessence. In Quintessence, the domination of the potential gave rise to the cosmological constant limit.
- **Case III: V(ϕ) $\neq 0$, λ and δ are constants.**
  - If we avoid the Phantom Field scenario, $\sigma < 0$.
  - There are two fixed points, both of which can be stable and unstable depending on λ and δ. For a stable fixed point to exist $\sigma > -3$. Point A is stable for $2\lambda < \delta + \sigma$ and Point B is stable for $2\lambda > \delta + \sigma$.
  - Point B is the Quintessence case. So we can get Quintessence as an end product of K-essence.
  - For both point A and B, e.o.s. $\gamma_\phi = -\frac{\sigma}{3}$. So whether the universe will continue to accelerate forever or whether it will start to decelerate depends on the slope of the potential. For $-3 < \sigma < -2$ the finals stage of universe is decelerated. For $-2 < \sigma < 0$, the universe will continue its accelerated expansion phase.
- **Case IV: V(ϕ) $\neq 0$, λ and δ are variables.**
  - For a stable fixed point to exist we must have $\sigma > -3$. To avoid phantom field we should have $\sigma < 0$.
  - A and B fixed points are unstable. C and D are stable for very restricted values of τ and Γ. Hence a small class of coefficients $\alpha(\phi)$ and $\beta(\phi)$ are allowed.
  - E is stable fixed point without any further condition on τ and Γ, i.e. no further constraints on the form of the coefficients. It also is the case of Quintessence since $y = 0$. So, Quintessence comes as a fate of the K-essence model.
  - The equation of state and scalar energy density parameter conditions are the same as Case III.

Reference

[1] C. Armendariz-Picon, T. Damour, and V. Mukhanov, Phys. Lett. B 458, (1999) 209. J. Garriga and V. Mukhanov, Phys. Lett. B 458, 219 (1999).
[2] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000).