Dynamical System Analysis of Quintessence Models

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MS Thesis Report I

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November 19, 2016

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Contents

1 Introduction

Recent observations suggest that the universe is in the state of accelerated expansion[1]-[2]. The matter dominated universe model fails to explain this. The Λ-CDM model can give rise to accelerated expansion but the value of the cosmological constant (Λ) to match the current energy density of the universe and the value obtained from high energy physics calculations has huge discrepancy[3]. To solve this, various models have been proposed with a scalar field which contributes negative pressure in the Friedmann equations[4]-[5]. This scalar field models are called dark energy models and the substance to produce this effect is called dark energy which is yet to be detected. The scalar field potentials determine the evolution of the universe as a system. The potentials can be broadly categorized into two types, Tracking and Thawing[10]-[12]. For tracking potentials, the equation of state (e.o.s.) of the universe gradually decreases to its asymptotic limit -1 (e.o.s. for Λ). On the other hand, the thawing potentials suggest that the e.o.s. of the universe is increasing very slowly from -1 following a slow-roll potential. The advantage of both of these models are that, they both bypasses the fine-tuning problem and co-incidence problem[6] by choosing a group of potentials and allowing a broad range of initial conditions, not a particular one. This helps in avoiding the choice of a precise initial condition that will give rise to current universe energy density. But none of these models has yet been hailed as the correct or preferred. However, if one can identify the correct model based on the recent observational data and predict the correct redshift value (z) when the expansion started from the model, it will be a huge advancement and give a boost to the dark energy model. The current objective of this project is to try to determine the preferred model using stability analysis tools for a dynamic system where the system is the universe.

2 The FRW metric and Friedmann equations

The FRW metric is taken here to find out Einstein's equations with a spatial curvature k=0 and also the homogeneity and isotropy of universe is considered (cosmological principle). The FRW metric is given by

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + sin^2\theta\, d\phi^2)] \tag{1}$$

where a(t) is the scale factor. Using this metric, and the stress-energy tensor for the perfect fluid,

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + P g_{\mu\nu} \tag{2}$$

we get the Friedmann equations from the Einstein Equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ as,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3\mathrm{p} + \rho) \tag{4}$$

3 Scalar field model

The Friedmann equations with known matter cannot explain the accelerated expansion since in eqn(3) both pressure and energy density is non-negative for any known matter. Hence $\ddot{a}$ is always negative. So, we introduce a scalar field to produce a negative pressure contribution in the Friedmann equations. If this negative pressure contribution is sufficiently high then, $\ddot{a}$ can be positive giving rise to a accelerated expansion.

For this, we introduce a Lagrangian corresponding to the scalar field with the Einstein- Hilbert Lagrangian.

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{5}$$

Taking variation with respect to the metric $g^{\mu\nu}$ and comparing with perfect fluid equation we get the pressure and energy density of the scalar field, which are given by eqn(5) and (6) respectively. The pressure and energy for the scalar field is given by,

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \tag{6}$$

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{7}$$

Again, taking variation with respect to the scalar field we get the Klein Gordon equation for this scalar field (also sometimes called as the continuity equation) given by,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{8}$$

The corresponding equation of state is given by

$$w_\phi = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)} \tag{9}$$

Clearly we get the cosmological constant model for $V(\phi) >> \frac{\dot{\phi}^2}{2}$ as then w_ϕ tends to -1. The Friedmann equations for a universe containing both matter and this scalar field can be written as

$$H^2 = \frac{1}{3}(\rho_b + \rho_\phi) \tag{10a}$$

$$\dot{H} = -\frac{1}{2}(\rho_b + \rho_\phi + p_b + p_\phi) \tag{10b}$$

where H is the Hubble parameter, given by $H = \frac{\dot{a}}{a}$. Here $8\pi G$ has been taken equal to 1. From here on, this unit will be used throughout this discussion.

4 System of equations

In order to transform the cosmological equations into an autonomous dynamical system, new auxiliary variables are introduced[7]. The variables x, y, and λ are defined as

$$x = \frac{\dot{\phi}}{\sqrt{6}H} \tag{11a}$$

$$y = \sqrt{\frac{V}{3H^2}} \tag{11b}$$

$$\lambda = -\frac{1}{V}\frac{dV}{d\phi} \tag{11c}$$

The system of equations in terms of these newly defined variables are given by

$$x' = \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2}x(x^2 - y^2 - 1) \tag{12a}$$

$$y' = -\sqrt{\frac{3}{2}}\lambda xy + \frac{3}{2}y(1 + x^2 - y^2) \tag{12b}$$

$$\lambda' = -\sqrt{6}\lambda^2(\Gamma - 1)x \tag{12c}$$

where $\Gamma = V\frac{\frac{d^2V}{d\phi^2}}{(\frac{dV}{d\phi})^2}$. We introduce another new set of variables which are observable quantities and hence makes the phase-space diagrams more comprehensible[8].

$$\Omega_\phi = \frac{\rho_\phi}{3H^2} = x^2 + y^2 \tag{13a}$$

$$\gamma_\phi = 1 + w_\phi = \frac{2x^2}{x^2 + y^2} \tag{13b}$$

Using this redefined variables, the equations become,

$$\Omega'_\phi = 3(1 - \gamma_\phi)\Omega_\phi(1 - \Omega_\phi) \tag{14a}$$

$$\gamma'_\phi = (2 - \gamma_\phi)(-3\gamma_\phi + \lambda\sqrt{3\gamma_\phi\Omega_\phi}) \tag{14b}$$

$$\lambda' = -\sqrt{3\gamma_\phi\Omega_\phi}\lambda^2(\Gamma - 1) \tag{14c}$$

5 Thawing Model

The Thawing model states that the equation of state (w_ϕ) was -1 and it slowly increased to its current value. The corresponding potential is a slow rolling potential which satisfies the following slow roll approximations[9]:

$$\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 << 1 \tag{15}$$

$$\frac{1}{V}\left(\frac{d^2V}{d\phi^2}\right) << 1 \tag{16}$$

Though the slow roll approximations were derived for potentials causing inflation when only the scalar field was present (contrary to the case now, because now both matter and scalar field are present in the universe), in this case we only take the approximation conditions to group together different types of potentials with the slow roll behaviour which can model the thawing case.

The three evolution equations (the derivatives are taken with respect to ln(a)) are written below with $w_b = 0$ (for matter)

Assumptions

For this model the assumptions are made,[10]

- The first assumption is that $\gamma_\phi << 1$ since w_ϕ is very close to -1.
- The second assumption is that λ is approximately constant, i.e.,

$$\lambda = \lambda_0 = -\frac{1}{V}\frac{dV}{d\phi}\Big|_{\phi=\phi_0} \tag{17}$$

  Here λ_0 is the initial value of λ before the slow roll down the potential begins. This approximation follows from the slow roll approximation itself,

$$\frac{1}{V}\left(\frac{d^2V}{d\phi^2}\right) - \left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 << 1 \tag{18}$$

  which is basically the two slow roll approximation conditions put together. This can be rewritten as the condition

$$|\frac{\lambda'}{\lambda}| << 1 \tag{19}$$

  which yields that λ is almost constant.

- This condition can be achieved by two ways, by demanding either $\Gamma \approx 1$ or λ to be small.

These assumptions indicate that the equation of state for the scalar field was frozen at -1 for a long time until Hubble expansion made the energy density of the scalar field and the matter comparable. The scalar field then began to thaw down the very flat potential and w_ϕ starts increasing slowly. Hence the value of w_ϕ is close to -1.

Approximated equations

The equations 14(a)-(c), with the above mentioned assumptions become,

$$\Omega'_\phi = 3\Omega_\phi(1-\Omega_\phi) \tag{20a}$$

$$\gamma'_\phi = -6\gamma_\phi + 2\lambda\sqrt{3\gamma_\phi\Omega_\phi} \tag{20b}$$

with λ = constant.
The first equation is an uncoupled equation and can be solved easily. We can find γ_ϕ as a function of Ω_ϕ and the scale factor a as well (Fig. 1).

Jacobian and Fixed points

The Jacobian of the system of equations is given by

$$J = \begin{pmatrix} 3(1-2\Omega_\phi) & 0 \\ \lambda\sqrt{\frac{3\gamma_\phi}{\Omega_\phi}} & -6+\lambda\sqrt{\frac{3\Omega_\phi}{\gamma_\phi}} \end{pmatrix} \tag{21}$$

The corresponding fixed points are

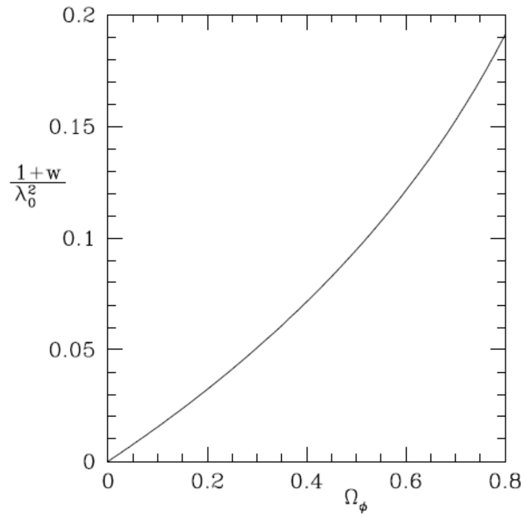


Figure 1: The dependence of γ_ϕ/λ_0^2 on Ω_ϕ for a nearly flat potential. λ_0 is the initial value of λ. (Source: R. Scherrer, A.A. Sen, Phys. Rev. D77, 083515, 2008.)

γ_ϕ	Ω_ϕ
0	0
0	1
$\lambda^2/3$	1

Now, we see that the second and third fixed points in the table are same as the case without any assumptions. The first fixed point is of interest because it gives rise to a singular Jacobian, as can be seen by putting $\gamma_\phi = 0$, and $\Omega_\phi = 0$ in the Jacobian matrix. However, when we plot the trajectory we see a saddle at (0,0).

- The physical significance of the fixed point (0,0) is that both effective equation of state and scalar field energy density parameter (Ω_ϕ) can be zero at the beginning of the universe when $H = \frac{\dot{a}}{a}$ (the Hubble parameter) value was infinity.
- When we extrapolate the system of equation backwards in time, we see that the system approaches the saddle.
- The system approaches the fixed point $(1, \lambda^2/3)$ when simulated forward in time (Fig. 2). It is a stable fixed point as can be verified by putting the fixed point values in the Jacobian. To be specific, it is a degenerate node with only one eigen-direction.
- The (0,0) fixed point can be justified as a saddle point if we make the transformation $\gamma_\phi = \alpha^2$. Then it is easy to check that the Jacobian yields a saddle point (as det(J)<0).
- Instead of evolving the system backwards and try to approach the point (0,0), we start with an initial condition sufficiently close to zero and integrate the system (Fig.3). We see that, after a perturbation from the point (0,0), the effective

equation of state rapidly comes down to a value very close to zero and then starts increasing very slowly (due to the nearly flat potential) and finally reaches the fixed point (1,$\lambda^2/3$). The rapid coming down of w_ϕ to -1 is somewhat reminiscent of inflation but during inflation there was no matter distribution in the universe.

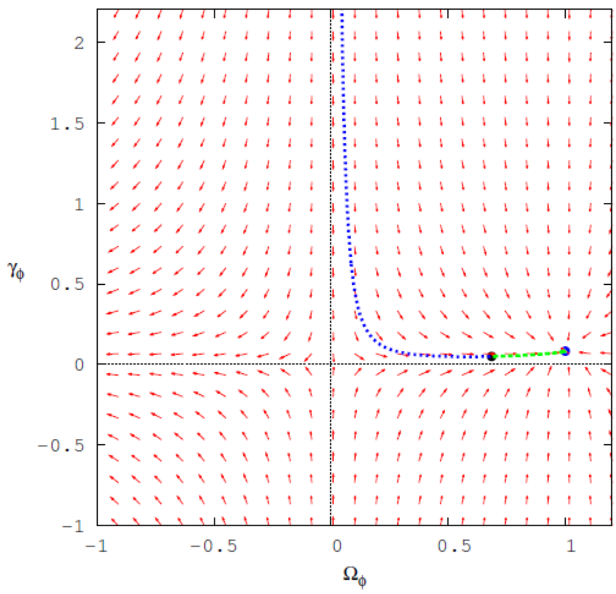


Figure 2: Trajectory for a thawing potential with $\lambda_0^2 = 0.25$

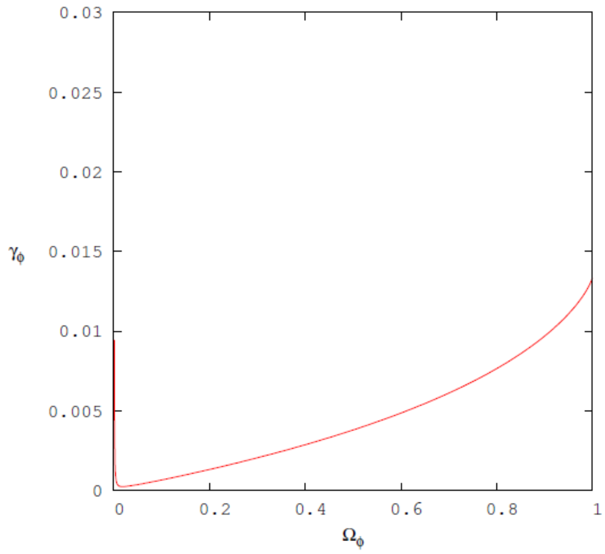


Figure 3: Perturbation from the fixed point (0,0) with $\lambda_0 = 0.2$

Thawing model in 3D

Here we consider the whole 3D model without the assumption that λ is a constant and let λ evolve as well.

- **For $\Gamma \approx 1$**
  The fixed points are (0,0,0), (1,0,0), and $(1,\lambda^2/3,\lambda)$. Out of these, the point (0,0,0) is unstable in all three directions. The fixed point (1,0,0) is attracting in the Ω_ϕ and γ_ϕ direction but the λ direction is unstable. The other fixed point or we should say set of fixed points are attracting. The evolution in this case is quasi-static, i.e., as λ increases with time, γ_ϕ goes to a new fixed point $(\lambda^2/3)$.

- **For $\Gamma \sim \mathcal{O}(1)$ or bigger**
  In this case all the fixed points are unstable and the $(1,\lambda^2/3,\lambda)$ fixed point does not exist.

- All the stability analysis is done by evolving a small perturbation from the fixed points. The diagrams for the first case is shown in the following figures (Fig.4 and Fig.5).

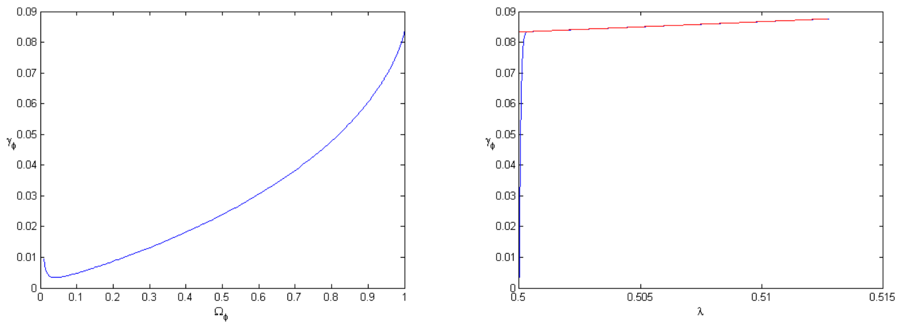


Figure 4: γ_ϕ vs. Ω_ϕ and γ_ϕ vs. λ plot. The red line in the second plot indicates $\lambda^2/3$ line.

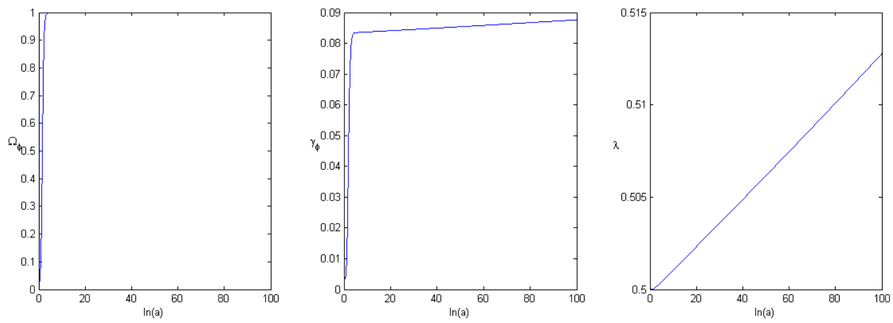


Figure 5: Time series plots of the 3 variables

6 Tracking Model

The tracking model[11] suggests that the equation of state w_ϕ is decreasing gradually and have reached its current value which is close to -1. It is called 'Tracking' because the energy density of the scalar field tracks the background energy density. In the radiation dominated era $\Omega_\phi < \Omega_m < \Omega_r$, and in the matter dominated era $\Omega_r < \Omega_\phi < \Omega_m$ and in this epoch, the scalar field energy density finally began to dominate giving rise to accelerated expansion. This is because the scalar field energy density falls at a slower rate than matter and radiation.
However the name tracking has another significance. It is insensitive to initial scalar field energy density upto a order of 100 magnitude. That is all the trajectories starting within this range follows a trajectory and finally converges to a solution called 'tracker solution'. Thus only the tracker solution which is a representative of the set of solutions for a given potential matters, and the initial information does not matter. Thus it evades the fine-tuning problem.
Tracking is exactly the opposite case of thawing. Here w_ϕ starts from a value higher than -1 and then evolves down the trajectory following the tracker solution and finally gets frozen at a value very very close to -1. So, sometimes it is also called 'Freezing model'.

Assumptions

The assumptions for this model are, [12]

- $\dot{\gamma_\phi}$ is negligible, i.e. the e.o.s. varies very slowly with time.
- $\Gamma \geq 1$ for tracking solution. This condition is needed to satisfy the following approximated equation (after the first assumption)

$$\frac{V'}{V} \sim \frac{1}{\sqrt{\Omega_\phi}} \tag{22}$$

  What this condition basically implies, is that $|\frac{V'}{V}|$ decreases as V decreases.
- Γ is almost constant w.r.t. time. This is needed for the converging behaviour of the tracker solutions.
- $\Gamma \approx 1$ for 'near tracking' solutions.

System of equations

The system of equation for Tracking model is same almost same as equation (14) except equation 14(c) which yields $\lambda' \approx 0$. So we consider λ = constant.

Fixed points

The fixed points of the system along with the stability is given by,

γ_ϕ	Ω_ϕ	Existence	Stability
2	1	$\forall\ \lambda$	unstable if $\lambda \geq -\sqrt{6}$ saddle if $\lambda < -\sqrt{6}$
1	$\frac{3}{\lambda^2}$	$\lambda^2 \geq 3$	Stable
$\frac{\lambda^2}{3}$	1	$\lambda^2 < 6$	stable if $\lambda^2 < 3$ saddle if $3 \leq \lambda^2 < 6$

- The (1,2) point is either a saddle or an unstable fixed point. Hence it is considered as the point where universe has started evolving.

- The fixed point (1,$\lambda^2/3$) is the same one obtained in the thawing case (Fig. 6(a)). This is a stable fixed point for $\lambda^2 < 3$, which is always the case for thawing but not for tracking. For tracking, the stable fixed point is decided depending upon the value of λ.

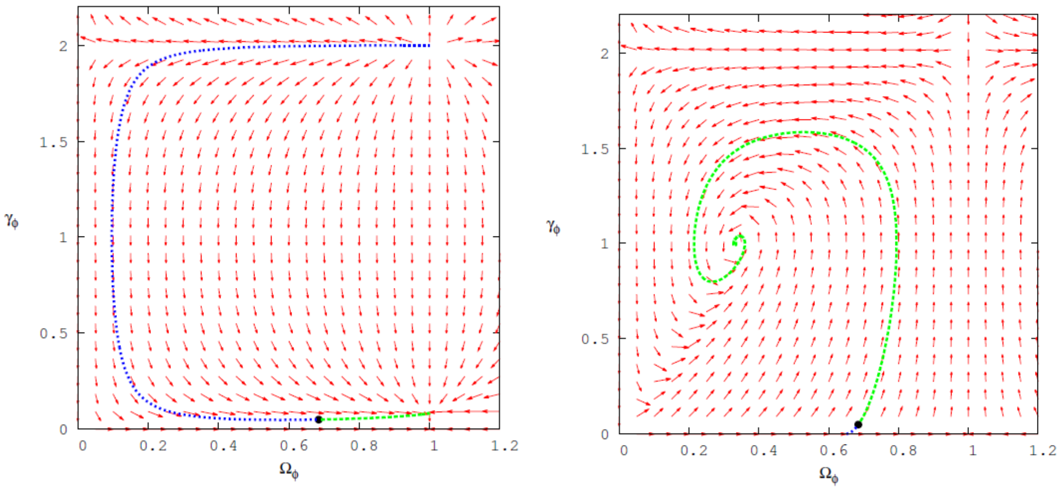
- For $\lambda^2 \geq 3$, the stable fixed point is $(3/\lambda^2, 1)$. For this case, the system cannot be backtracked in time to the fixed point (1,2). Also, depending upon the values of λ the trajectory could be spiral. Numerical calculation show that for $\lambda^2 > 3.4$ the trajectory will be a spiral (Fig. 6(b)).

- The spiral is affected by the influence of the fixed point (1,2). If the initial condition is given to be the current values of γ_ϕ and Ω_ϕ, the spiral never crosses the line $\gamma_\phi = 2/3$ more than thrice (the line $\gamma_\phi = 2/3$ is important because for $\gamma_\phi < 2/3$, we get accelerated expansion). This means that after the current phase of accelerated expansion there can only be another period of accelerated expansion before the universe settles down to the fixed point with no acceleration (Fig.7).

- After the current phase of acceleration, whether there will be another phase of acceleration depends upon the values of λ. For $\lambda \geq 3.57$ we get another phase of acceleration.

- As λ is increased, γ_ϕ tends to 2. The trajectory gradually converges with the manifolds of the saddle point (1,2).

- For $\lambda^2 < 3$, the stable fixed point ($3/\lambda^2$,1) disappears and we get a new stable fixed point (1,$\lambda^2/3$). Trajectories for these values of λ can be backtracked in time to the fixed point (1,2). So, this fixed point is preferred to the previous one. This fixed point also appears in the thawing case, though in that case it was the only fixed point for the system.



(a) $\lambda = 0.5$

(b) $\lambda = 3$

Figure 6: Phase space plot for two different values of λ. The black dot represents the current value of Ω_ϕ and γ_ϕ

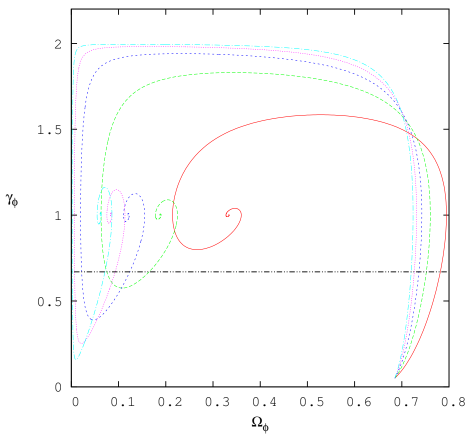


Figure 7: Trajectory for various λ values (from λ = 3-7). As λ increases the trajectory approaches the $\gamma_\phi = 2$ line.

7 Conclusion and Future Work

In conclusion we can say,

- Thawing and Tracking both gives a fixed point $(1,\lambda^2/3)$.
- In thawing, the past cannot be tracked to a fixed point exactly but in tracking, it depends on λ.

- However, in thawing the (0,0) point is a saddle and can be interpreted as a singularity at the beginning of the universe. A small perturbation shows inflation like behaviour at first. However since our system is has both scalar field and matter, drawing any conclusion from this similarity would be naive. Further investigation on this needed and is in the future scope of this project.
- The dynamical system analysis does not clearly favour one of these two models which was our primary aim. However further investigation on the time-scale of the evolution and start of accelerated expansion needs to be done in order be conclusive on this topic. This is also a scope of this project.

Acknowledgement

I would like to thank my supervisor, Dr. Anandamohan Ghosh and Prof. Narayan Banerjee for guiding and supporting me throughout this work. Their invaluable insights and suggestions has helped me understand a lot in this area of physics. I would also like to thank Dr. Golam Mortuza Hossain for valuable suggestions. I also express my gratitude towards some of my friends at DPS for their help in this matter.

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