Probabilistic Analysis of Stochastic Processes

Towards a theory that applies to any general Gaussian process

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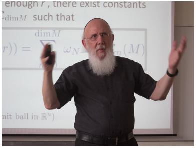

Prof. Robert J Adler

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Abstract

- Gaussian Processes are one of the most important families of stochastic processes, with uses across multiple disciplines.
- In the late 1960s - no need for finer geometric details of the parameter space
- 'Entropy'
- Entropy arguments to establish a.s sample path continuity + Applications
- Essential inequalities.

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Motivation for a 'Modern Theory'

Context:- Almost all existing literature establishes sufficient conditions for sample path continuity of Gaussian Processes separately for those defined on the real line, those on multidimensional parameter spaces, those that are function-indexed, set-indexed and so on.

Aim: To develop a unifying theory that does not rely on the geometrical structure of the parameter space, and thus applies to the analysis of all kinds of Gaussian Processes.

Two problems of focus: **Sample Path continuity** and **Distribution of the supremum of a Gaussian Process over a fixed subset of its parameter**

Preliminaries

- $\{X_t | t \in T\}$ such that $\forall t_i \in T, \alpha_{t_i} \in \mathbb{R}$; $\sum_i \alpha_{t_i} X_{t_i}$ is a centered Gaussian random variable, defined on a complete probability space, $(\Omega, \mathscr{F}, \mathcal{P})$.
- The distribution of the centered Gaussian Process described above is completely characterized by the positive semi-definite covariance function R defined on T such that R(s,t) = E$(W_s W_t)$.
- No assumptions are made on the specific geometrical structure of the parameter space, T. In general, it will only be required that T is a totally bounded metric space. It is also assumed, unless stated otherwise, that $\{X_t\}_{t \in T}$ is a separable stochastic process (for ease of analysis).
- Given t $\in$ T, X is said to be almost sure continuous at t if P$(\{\omega \in \Omega | lim_{s \to t} | X_s(\omega) - X_t(\omega)| = 0\})$=1.

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More than a Handful

$\mathbb{N}/\mathbb{Z}/\mathbb{Q} \to \mathbb{R} \to \mathbb{C} \to \mathbb{R}^2 \to \mathfrak{F}^{(q)} \to$ Measure-indexed Processes, Set-indexed, and so on.
Only assuming a centred Gaussian process defined on a complete probability space, and on a totally bounded, separable parameter space...

Background

Define $\rho_\tau^2(u) = Sup_{\tau(s,t)\leq u} E(X_s - X_t)^2$. Before investigating a.s. continuity, we at least need $\rho_\tau^2(u) \to 0$ as u $\to$ 0.
Assuming we have mean square continuity (iff R is continuous), it is observed that for almost sure continuity, only the the rate of convergence of ρ remains the additional concern.

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The Canonical Metric

Defining a metric easier to work with.

Definition

Define d:T x T such that d(s,t) $= [E(X_s - X_t)^2]^{(\frac{1}{2})}$. This naturally induces a pseudo-metric in T through the covariance function of X_t, circumventing any dependence on the geometry of T. This d is called the *canonical metric*.

Equivalence of d-continuity and τ-continuity

If X is defined on a totally bounded and complete (hence, compact) metric space, (T,τ) with a continuous covariance function R; then τ-continuity and d-continuity of X are equivalent.
The last result means that we can work with this metric under some minimal conditions on the parameter space. Note that this induces a metric structure through the covariance function even in parameter spaces without an inherent metric structure.

Notice that $\rho_d^2(u)$ loses information. We need to tap into the information about the process from d. How do we capture it?

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The First 'General Theory' Result

We capture the information contained in how the canonical metric measures T and subsets of T, through the concept of Entropy.

Definition

Let N(ϵ) be the least number of d-balls of radius ϵ required to cover (T,d). Then, H(ϵ) = log(N(ϵ) is called the Metric Entropy Function for T (or X).

Note: N(ϵ) $< \infty \; \forall \; \epsilon > 0$ is assumed to be true, ie, T is totally bounded in the canonical metric. It is a weak assumption satisfied by most general Gaussian processes.

Main Result

If X is a centered Gaussian Process on a totally bounded parameter space T, and if the integral of the metric entropy over [0,∞), $\int_0^\infty [log(N(\epsilon))]^{\frac{1}{2}} d\epsilon < \infty$; then X is almost surely continuous on T. ....(1)

Remark: The upper limit of the integral can be replaced with diam(T).

The Brownian Family (Recap)

- Given a σ-finite (E, ε, ν), a GWN based on ν is a random set function W on ε such that W(A) is centred, $EW^2(A) < \infty$, W has independent increments, and finite additivity. $R_\nu(A \times B) = EW(A)W(B) = \nu(A \cap B)$
- $\{W((0,t])|t \in \mathbb{R}_k^+\}$ is called the Brownian Sheet Process on $\mathbb{R}_+^k$. The Brownian Sheet processes are central to the k-dimensional central limit theorem. IID processes on the k-dimensional integer lattice in $\mathbb{R}_+^k$ converge weakly to the Brownian Sheet.
- $W_t' = W_t - |t|W_1$ is called the pinned Brownian sheet. Empirical cdfs and measures built from iid random variables uniformly distributed on $[0,1]^k$ also converge to the pinned Brownian Sheet defined over appropriate spaces.
- **Proposition 1**: The Brownian Sheet on $[0,1]^k$ is continuous.
- Consider a partial order on $\mathbb{R}_k^+$ such that $s \leq t \implies s_i \leq t_i \ \forall i \in \mathbb{N}_k$ and similarly for '<'. A set A $\subset \mathbb{R}_+^k$ is said to be a **lower layer** if for any two points s,t $\in \mathbb{R}_k^+$; s$\leq$t and t$\in$A $\implies$ s$\in$A.
- **Proposition 2**: The Brownian Sheet on the lower layers in $[0,1]^2$ is discontinuous and unbounded with probability 1.
- Also saw that W is unbounded over the convex subsets of $[0,1]^3$ but continuous over those of $[0,1]^2$.

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Other Gaussian Processes (Recap)

- **On $\mathbb{R}$**: It is seen that relying on the parameter space unnecessarily complicates the analysis. However, as expected from a specialization, some processes are not characterized as continuous directly from the integral condition here.
- It can be shown that continuity is a consequence of the smoothness of the covariance function at the origin.
- **Random Fields**: Appropriate changes to the result for processes on $\mathbb{R}$.
- **Gaussian Fourier Series**: Uniform convergence of Gaussian Fourier Series are important in non-random harmonic analysis. Entropy arguments play an important role in simplifying analysis of such series.
- **Talagrand Expansion**:Let X be a centered Gaussian process on a compact metric space T. Then X is continuous $\iff$ X has a continuous covariance function and there exists a centered Gaussian sequence $\{Y_n\}$ with variances $\sigma^2(Y_n)$ such that $\forall t, X_t := \sum_{n=0}^{\infty} \alpha_n(t) Y_n$ with the following conditions satisfied:
  - $lim_{n \longrightarrow \infty} (log(n))^{\frac{1}{2}} \sigma(Y_n) = 0$
  - For each t $\in$ T, $\alpha_n(t) \geq 0 and \sum_{n=0}^{\infty} \alpha_n(t) \leq 1$.
- **Generalised Random Fields**: Useful in study of infinite dimensional diffusions, stochastic PDEs, Quantum Field Theory, etc.

Set-indexed Processes

Dudley Class of Sets

- Understanding set indexed processes is motivated by their role in the development of multivariate 'Kolmogorov-Smirnov Tests'.
- Take std. atlas $\{(V_j, F_j)\}$ on S^k. Define $\mathfrak{F}^{(q)}(S^k, M) =$ Set of all $\mathbb{R}$-valued functions ϕ such that the composition of its restriction on a coordinate chart $\phi|_{V_j} \circ F_j \in \mathfrak{F}^{(q)}(B^k, M, ..., M)$. Consider $D(k+1, q, M) = \Pi_{i=1}^{k+1} \mathfrak{F}^{(q)}(S^k, M)$.
- These families correspond to sets, on which one more alg. geometric construction gives rise to the 'Dudley sets'.
- Defining a Brownian sheet based on Lebesgue measure appropriately on them gives a nice result that imposes conditions on q and k, to determine continuity/unboundedness.

Set-indexed Processes

The Vapnik-Cervonenkis (VC) Class

- What if we move onto Gaussian White Noises defined with more general σ-finite measure spaces?
- Let E $\subset \mathbb{R}^k$ and ν be a probability measure on E. Given a class $\mathfrak{C}$ of subsets of E and a finite set F$\subset$E, let $\triangle^{\mathfrak{C}}$(F) be the number of different sets $C \cap F$ for $C \in \mathfrak{C}$.
- For n=1,2,..., let $m^{\mathfrak{C}}(n) :=$ max$\{\triangle^{\mathfrak{C}}(F)$:F has n elements$\}$.

$$\text{Set } V(\mathfrak{C}) = \begin{cases} inf\{n : m^{C}(n) < 2^n\} & m^{\mathfrak{C}}(n) < 2^n \text{ for some n} \\ \infty & m^{\mathfrak{C}}(n) = 2^n \text{ for all n} \end{cases}$$

- The class $\mathfrak{C}$ is called a VC class if $m^{\mathfrak{C}}(n) < 2^n$ for some n, ie, if V($\mathfrak{C}$) $< \infty$. The number V($\mathfrak{C}$) is called the VC index of $\mathfrak{C}$.
- W is a GWN based on prob. measure ν on some (E, ε, ν).$\mathfrak{A}$ is a VC class of sets in ε. Then, there exists a nice bound for the entropy function, that depends on the VC class and ε.
- Corollary: W is continuous over $\mathfrak{A}$.

Other Applications

Vector Valued Processes

For an $\mathbb{R}^N$-valued process, analyse continuity of each component process separately...

Non-Gaussian Processes

- Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function and $Y_t = F(X_t)$ where X is a Gaussian process...
- If $X_1(t), ..., X_N(t)$ are independent Gaussian processes defined on same probability space, χ^2-process, $Z_t = \sum_{i=1}^{N} \chi_i^2(t)$, ...
- There is a result by Pisier and Fernique, that uses the entropy arguments for Gaussian processes to Banach-space valued processes (under some minimal conditions again) unrelated to any Gaussian process.

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Motivation

- Characterizing the supremum of a Gaussian process is an important aspect of analysis in Gaussian Process Theory, apart from determining boundedness of the process.
- Statistical tests and approximations rely on some limit distribution being the supremum of a Gaussian process.

Example (Kolmogov-Smirnov Test)

Let $\{X_i\}_{i=1}^n$ be an iid sample. Let $F_n(x) = \sum_{i=1}^n \mathbb{1}_{(-\infty,x]}(X_i)$ be the empirical distribution function. Then, $D_n(F) = \underset{x}{sup}|F_n(x) - F(x)|$ is called the Kolmogorov-Smirnov Statistic for a given cdf F(x). This a.s goes to zero as n goes to ∞, but upon normalization, $\sqrt{n}D_n \overset{n\to\infty}{\to} \underset{t}{sup}|B(F(t))|$ under the null hypothesis that the sample comes from F(x), where B is the Brownian Bridge.

- Probabilistic Questions surrounding 'hitting times' of a process require analysis of suprema and infima.

Behaviour of $Sup_T X_t$

- P(λ)=P$\{Sup_{t\in T} X_t \geq \lambda\}$ is known only for six specific stationary processes, defined on a finite interval in $\mathbb{R}$.
- Asymptotic behaviour of P(λ) has a reasonably full theory. The behaviour of P(λ) as $\lambda \to \infty$ depends on:-
  1. Lack of Homogenity of X on T.
  2. Local smoothness or lack thereof.

Result 0

If X is a centered Gaussian rv with variance σ^2, a clever integration by parts yields

$$\frac{(1-\frac{\sigma^2}{\lambda^2})(\frac{\sigma}{\sqrt{2\pi}})}{\lambda} e^{-\frac{\lambda^2}{2\sigma^2}} \leq P\{X > \lambda\} \leq \frac{(\frac{\sigma}{\sqrt{2\pi}})}{\lambda} e^{-\frac{\lambda^2}{2\sigma^2}} \tag{1}$$

$$\overset{\text{limit } \lambda\to\infty}{\Longrightarrow} lim_{\lambda\to\infty} \frac{logP\{X > \lambda\}}{\lambda^2} = -\frac{1}{2\sigma^2} \tag{2}$$

Borell's Inequality

Result 1 (Landau and Shepp(1970) and Marcus and Shepp (1971))

If $\{X_t\}_{t\in T}$ has bounded sample paths w.p. 1, then

$$lim_{\lambda\to\infty}\frac{logP\{Sup_{t\in T}X_t>\lambda\}}{\lambda^2}=\frac{-1}{2\sigma_T^2} \text{ where } \sigma_T^2=Sup_{t\in T}EX_t^2 \tag{3}$$

- Note that equations (2) and (3) seem to indicate that the asymptotic distribution of $Sup_TX_t \overset{d}{\sim} X\sim N(0,\sigma_T^2)$.
- The strongest form of Result 1 is due to Borell.

Theorem 2: Borell's Inequality (The Maurey-Pisier (1986) Version)

Let $\{X_t\}_{t\in T}$ be a centred Gaussian process with sample paths bounded a.s. Let $||X||=Sup_{t\in T}X_t$. Then, E$||X||<\infty$ and for all $\lambda>0$;

$P\{\| \, ||X||-E||X|| \, |>\lambda\}\leq 2e^{\frac{-\lambda^2}{2\sigma_T^2}}$. Hence, $\forall\lambda>E||X||; P\{||X||>\lambda\}\leq 2e^{\frac{-(\lambda-E||X||)^2}{2\sigma_T^2}}$

Remarks

- Take a λ large enough. Use the last inequality of Theorem 2, and applying logarithm, limit will give (2).
- $E||X||$ is required to be known to make any use of this theorem.
- Borell's inequality also helps find bound on $Sup_{t \in T} \; |X_t|$. P$\{Sup_{t \in T}|X_t| > \lambda\}$ $= 2$P$\{||X|| > \lambda\}$. (Just use symmetry)

Lemma

Let f: $\mathbb{R}^k \to \mathbb{R}$ *have derivatives of up to second order bounded pointwise by* $Ae^{B||x||}$ *for some* A, $B < \infty$. *Let* $X \sim N_k(0, V)$. *If* $|f(x) - f(y)| \leq ||x - y||$ *for all* x, $y \in \mathbb{R}^k$, *then for all* $\lambda > 0$,

$$P\{|f(X) - Ef(X)| > \lambda\} \leq 2e^{\frac{-\lambda^2}{2\sigma^2}} \text{ where } \sigma^2 = Sup_{1 \leq i \leq k} V(i,i) = Sup_{1 \leq i \leq k} EX_i^2$$

Proof of Theorem 2.

- Assume $|T| < \infty$, let T=$\{t_1, ..., t_k\}$. Then, E$[\max(X_{t_1}, ..., X_{t_k})] < \infty$ trivially. Now, we need an appropriate $f \in C^2(\mathbb{R}^k)$ that approximates $\max(x_1, ..., x_k)$ to use the above Lemma. This is done with an appropriate convolution. We get the result for finite T then.

Proof of Theorem 2.

- For any general T, assume $\mathrm{E}||X|| = \infty$. Choose a $\lambda_0 > 0$ such that $\mathrm{P}\{||X|| < \lambda_0\} \geq \frac{3}{4}$ and $e^{\frac{-\lambda_0^2}{\sigma_t^2}} \leq \frac{1}{4}$. By MCT, $E||X||_{T_n} \to E||X||_T$ as $n \to \infty$. Then, $\exists N \in \mathbb{N}$ st $\forall n \geq N, E||X||_{T_n} > 2\lambda_0$. Then, we can show that
  $P\{|\ ||X||_{T_n} - E||X||_{T_n}\ | > \lambda_0\} \leq 2e^{\frac{-\lambda^2}{2\sigma_T^2}} \leq \frac{1}{2}$ and
  $P\{|\ ||X||_{T_n} - E||X||_{T_n}\ | > \lambda_0\} \geq P\{E||X||_{T_n} - ||X||_{T_n} > \lambda_0\} \geq P(||X||_T < E||X||_{T_n} - \lambda_0) \geq P(||X||_T < \lambda_0) \geq \frac{3}{4}, (\rightarrow\leftarrow) \implies E||X||_T < \infty$.
- Extend finite case to countable set: For any countable set T=$\{t_1, t_2, ...\}$, $||X||_{T_n} \overset{a.s}{\to} ||X||_T$ as $n \to \infty$, (use the fact that $\underset{T}{Sup} X_t < \infty$, definitions of sup, max, and it follows) where $||X||_{T_n} = max_{\{t_1,...,t_n\}}(X_{t_1}, ..., X_{t_n})$ for each n $\in \mathbb{N}$.
- By MCT, $E||X||_{T_n} \to E||X||_T$, and $\sigma_{T_n}^2 \to \sigma_T^2$ as $n \to \infty$
- Then, $K_n = |||X||_{T_n} - E||X||_{T_n} \overset{a.s}{\to} K = |||X|| - E||X|||$ as $n \to \infty$. By Fatou's Lemma,

$$P\{\omega | K(\omega) > \lambda\} = \int_\Omega \mathbb{1}\{K(\omega) > \lambda\} d\mathcal{P} \leq \liminf_{n\to\infty} P(\omega | K_n(\omega) > \lambda) \leq 2e^{\frac{-\lambda^2}{2\sigma_T^2}}$$

Kahane's Inequality

- Now, use separability (initial assumptions) of T to extend to any general separable parameter space T from the countable case, since supremum over a countably dense subset is equal to supremum over the entire set (easy to check with definition of sup, dense property and ϵ-forcing principle).

Theorem 3 - Kahane's Inequality

Let X and Y be a.s bounded centred Gaussian random vectors on $\mathbb{R}^n$. Let $f \in C^2(\mathbb{R}^n)$ be such that its partial derivatives up to 2nd order have subgaussian growth. Also, $EY_iY_j \lesseqgtr EX_iX_j \implies \frac{\partial^2 f}{\partial x_i \partial x_j} \lesseqgtr 0$. Then, Ef(X) $\leq$ Ef(Y).

Proof.

- Take independent copies on same probability space. Interpolate Z$=cos\theta X + sin\theta$Y. Prove $\Psi'(\theta) > 0$ where $\Psi(\theta) = Ef(Z(\theta))$ (Subgaussian growth allows DCT-use. Then, use Gaussian integration by parts)

□

Slepian's Inequality

Corollary: Slepian's Inequality

If X,Y are a.s bounded centred Gaussian processes on T such that $EX_t^2 = EY_t^2$ for all $t \in T$ and $E(X_t - X_s)^2 \leq E(Y_t - Y_s)^2$ for all s,t $\in T$; then for all real λ, P$\{||X|| > \lambda\} \leq P\{||Y|| > \lambda\}$.

Overview of Proof of Slepian's Inequality.

- Take $|T| = k$. Consider any collection of non-negative bounded $\mathbb{R}$-valued functions on $\mathbb{R}$ that are smooth and non-increasing, $f_1, f_2, ..., f_k$. Define h(x)=$\prod_{i=1}^{k} f_i(x_i)$. Then, $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for each $i \neq j$. Then, verify requirements of Theorem 3 and apply it to get $Eh(Y) \leq Eh(X)$. Now each i, take the appropriate collection $\{f_i^{(n)}\}_n$ such that $\lim_{n \downarrow \infty} f_i^{(n)} = \mathbb{1}_{(-\infty,\lambda]}$. Then, applying DCT to interchange an appropriate limit and expectation, we get the required result for finite T case.
- Now, extend to countable T by taking limit, and applying Fatou's Lemma and DCT.
- Finally, argue the result is true for general (separable) T using a countable

Stronger Result

Corollary of Slepian's Inequality: $E||X|| \leq E||Y||$ under the same conditions. Can we make this result stronger?

Lemma

X, Y - 2 centred Gaussian rvs $\implies Emax(X,Y) = \frac{\sqrt{E(X-Y)^2}}{\sqrt{2\pi}}$.

Proof of Lemma:

Note that if Z is a.s. bounded, centred Gaussian process on T, then $E(Sup_T(Z_t + Y)) = E||Z||$, and define process $Z_1 = X - Y, Z_2 = 0$. Done.

Result 3

If X, Y are a.s bounded, centred Gaussian processes on T such that $E(X_s - X_t)^2 \leq E(Y_s - Y_t)^2$; then $E||X|| \leq 2E||Y||$.

Idea of Proof of Result 3.

Fix t_0 and define $\alpha^2 = Sup_{t \in T} E(Y_t - Y_{t_0})^2$. Set $X'_t = X_t - X_{t_0}$ and Y'_t similarly. Then, define $\hat{X}_t = X'_t + \eta\alpha$, $\hat{Y}_t = Y'_t + \eta' g(t)$ where $\eta, \eta' \overset{iid}{\sim} N(0,1)$ and $g^2(t) = \alpha^2 - E(Y'_t)^2 + E(X'_t)^2$. Apply above corollary and lemma. □

Sudakov-Fernique Inequality - Going more stronger

Theorem 4

X, Y are a.s bounded, centred Gaussian processes on T such that $E(X_s - X_t)^2 \leq E(Y_s - Y_t)^2 \implies E||X|| \leq E||Y||$.

Proof.

- $|T| = k < \infty$: Define $f_\beta(x) = \frac{log(\sum_{i=1}^{k} e^{\beta x_i})}{\beta}$ for some $\beta > 0$. Clearly f$\in C^2(\mathbb{R})$ and f_β being a smooth approximation of max function, is integrable since X, Y are bounded a.s. Repeat the methods used in Kahane's inequality, to get $Ef_\beta(X) \leq Ef_\beta(Y)$. Take limit β to zero both sides and apply DCT, to get the desired result.
- Take general (separable) T. Let T'=$\{t_1, t_2, ...\}$ be a countable dense subset of T. Let $T_n = \{t_1, ..., t_n\}$ for each $n \in \mathbb{N}$. For each n, use the finite case and apply limit $n \to \infty$ both sides with MCT to get $E||X||_{T'} \leq E||Y||_{T'}$. Hence proved.

□

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Introduction

If X is a centred Gaussian process on a totally bounded and complete (ie, compact) parameter space, then a.s continuity $\implies$ $\text{Sup}_{t \in T} X_t$ is a.s bounded. Under what conditions is the converse true?

Lemma

X is a centered Gaussian process on T and $t_0 \in T$. Then,
$E sup_{t\in T} X_t \leq E sup_{t \in T}|X_t| \leq E|X_{t_0}| + 2E sup_{t \in T} X_t$.

This lemma essentially tells that boundedness of $Sup_{t\in T} X_t$ and $Sup_{t \in T}|X_t|$ are equivalent. If X is centred Gaussian process on T, then;

Theorem 5

$P\{Sup_{t \in T} X_t < \infty\} = 1 \iff E sup_{t\in T} X_t \iff E e^{\alpha ||X||^2} < \infty$ for small α

Proof.

$E e^{\alpha||X||^2} \leq \int_0^\infty P(||X|| > \lambda) d\lambda$. Break integral into 2 parts: 0 to $E||X||$, and $E||X||$ to ∞. First part $= E||X|| < \infty$, and Borell's inequality gives a bound for the second part which is finite if $\int_0^\infty u e^{\alpha u^2 - \frac{(u - E||X||)^2}{2\sigma_T^2}} du < \infty$. Hence follows. □

Relationship between Boundedness and Continuity

Theorem 6

X is a.s bounded on (T,τ). τ is a metric on T such that d is τ-uniformly continuous. Then, X is τ-uniformly continuous w.p. 1 $\iff$ $lim_{\eta\to\infty}\phi_\tau(\eta) = 0$, where ϕ_τ is given by $\phi_\tau(\eta) = E[Sup_{\tau(s,t)<\eta}(X_s - X_t)]$.

Corollary

Assume the conditions of Theorem 6, and that $lim_{\eta\to 0}\phi_\tau(\eta) = 0$. Then, for all $\epsilon > 0, \exists$ an a.s. finite random variable $\delta = \delta(\omega)$ such that, for almost all ω,

$$W_\tau(\eta) \leq \phi_\tau(\eta)|log\phi_\tau(\eta)|^\epsilon,$$

for all $\eta \leq \delta(\omega)$. That is, $\phi_\tau(.)|log\phi_\tau(.)|^\epsilon$ is a uniform sample modulus for X in the metric τ.

Corollary

Let X be as in Theorem 6, and for $t \in T$, $\phi^t_\tau(\eta) = Esup_{s:\tau(s,t)<\eta}(X_s - X_t)$. Then, X is a.s. continuous at iff $lim_{\eta\to 0}\phi^t_\tau(\eta) = 0$.

Zero-One Laws and Continuity

Gaussian process theory is apparently very rich in 'zero-one' laws. Though the proofs are highly involved, here are some interesting results that give a lot of information about the continuity question for any general Gaussian process!

Theorem 7

For a Gaussian process X on T, $P\{\lim_{s\to t} X_s = X_t \text{ for all } t \in T\} = 1 \iff P\{\lim_{s\to t} X_s = X_t\} = 1$, for each $t \in T$.

Theorem 8

P{X is continuous for all t $\in T\} = 0$ or 1.

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Summary

- A custom-tailored device to handle several kinds of Gaussian processes at once.
- Main Entropy Result, applications and implications.
- The supremum of a Gaussian process behaves like a normal random variable.
- Borell's Inequality- bounds the tail of suprema of Gaussian distributions, helps prove important results relating continuity and boundedness
- Kahane's Inequality- gives a nice comparison between two processes, with respect to expectations of certain functions. Helps prove several other inequalities including Slepian's, Sudakov-Fernique- that give a lot of information about the supremum too.
- Boundedness of supremum implies boundedness of exponential moments too under some conditions.
- Saw a nice result charactrizing continuity with boundedness of supremum and vice versa.
- The Zero-One Laws

References

- Introduction to Continuity and Extrema of General Gaussian Processes (1990) - RJ Adler
- Prof. Manjunath Krishnapur's notes on Comparison Inequalities

That's pretty much it. Thank you.