Nodal Discontinuous Galerkin Method to Model Gravitational Waves from Extreme-Mass-Ratio Black Hole Binaries

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Outline

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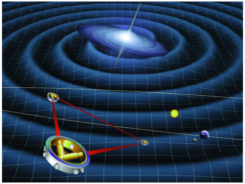
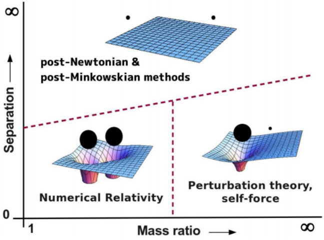
Black Holes and Laser Interferometer Space Antenna

- Various methods can be used to model sources for Gravitational Waves.





- The band at which LISA (0.1 mHz to 1Hz) will be sensitive, lies in the domain of perturbation theory.
- We use $s = 0$ and $T = 0$ Teukolsky formalism for modeling.

Teukolsky Equation

$$-\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 sin^2\theta\right]\partial_{tt}\Psi - \frac{4Mar}{\Delta}\partial_{t\phi}\Psi + \partial_r(\Delta\partial_r\Psi) + \frac{1}{sin}\partial_\theta(sin\theta\partial_\theta\Psi) + \left[\frac{1}{sin^2\theta} - \frac{a^2}{\Delta}\right]\partial_{\phi\phi}\Psi = 0$$

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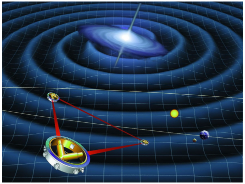

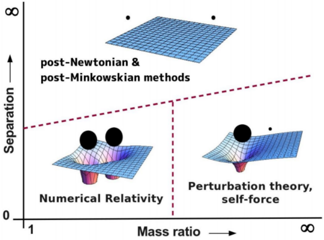

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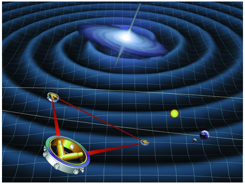

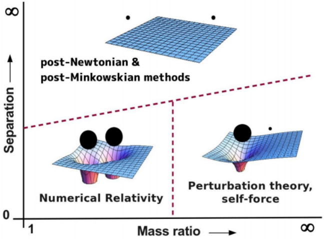

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Previous Works and Our Contribution

- We decompose the Teukolsky equation into a 1+1D system of equations. Dr. Burko and Dr. Khanna have also done a similar work before [1].
- We follow the same process but without the expansion of wave function.
- We end up with the following differential equation for Kerr space-time with mass M and spin parameter a:

Master Equation

$$-\ddot{\Psi}_L + \Psi_L'' + \frac{\Delta}{(r^2+a^2)^2}\left[\frac{3r^2\Delta}{(r^2+a^2)^2} - L(L+1) - \frac{r(2r-2M)}{(r^2+a^2)} - \frac{\Delta}{(a^2+r^2)}\right]\Psi_L =$$
$$-\frac{\Delta a^2}{(r^2+a^2)^2}\sum_{l=0}^{\infty} C_{lL}\ddot{\Psi}_l$$

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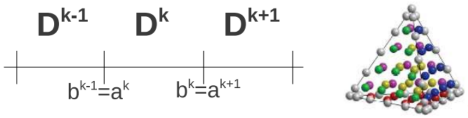
Outline

What is this dG method?

- The grid is unstructured. We choose lines, triangles, and tetrahedrons for 1D, 2D, and 3D respectively.



- The local solution is given as the sum of polynomials of degree at most N.

$$x \in D^k : \Psi^k(x,t) = \sum_{i=0}^{N} \Psi^k(x_i,t)\, l_i^k(x)$$

- The global solution is direct sum of local solutions:

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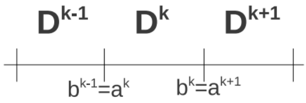

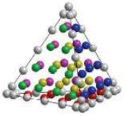

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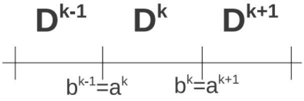

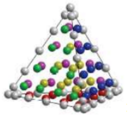

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Fluxes

- For a equation of form $L\Psi = \partial_t \Psi + \partial_x f(\Psi) + V\Psi = 0$ where Ψ and f are vectors, and V a matrix, the residual $L\Psi_h$ is integrated across all basis functions.
- IBPs are performed prior to coupling the subdomains to one.
- An example of flux could be central flux, $f^* = \frac{f(\Psi^+)+f(\Psi^-)}{2}$. It should be a function of Ψ^+ and Ψ^-.
- The flux is responsible for passing information between elements, implementing boundary conditions, and ensuring stability of scheme.

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dG method + δ

- The secondary black hole is represented on the grid as a singularity or a δ function.
- Let us consider a simple hyperbolic PDE with a δ source term:

$$\frac{1}{c}\partial_t\Psi + \partial_x\Psi = G(t)\delta(x)$$

- The δ term only shows up as an extra term in the flux. We add the required amount to the flux at the boundary.

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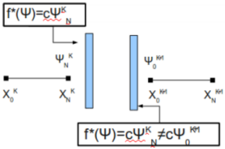

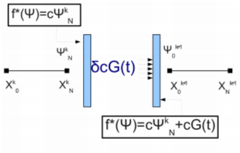

Why do we need dG method?

- Compared to previous methods like FD which uses narrow Gaussian, we effectively remove the particle hence no accuracy loss.
- The fields are smooth to the left and right of the particle
- Most importantly, long time evolution is needed to observe late time tails hence our high order method.
- Exact outgoing BCs and waveform extraction techniques.

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How everything relates together

- We introduce two auxiliary variables $\Pi = -\frac{\partial \Psi}{\partial t}$ and $\Phi = \frac{\partial \Psi}{\partial r^*}$ and after successful transformations and diagonalization ($W = -\Pi - \Phi$ and $X = -\Pi + \Phi$) the wave equations reduces to two copies of Advection equation.

$$
\begin{aligned}
\partial_t \Psi &= \frac{1}{2}(W + X) \\
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\partial_t X &= \partial_x X - V\Psi + \left(J_\Pi - J_\Phi\right)\delta\left(x - x_p\right)
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- We have used this method to solve the wave equation and some...

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Numerical Results

- We now have a useful numerical scheme. For sufficiently smooth solutions the error decays like

$$\left\|\Psi - \Psi_h^k\right\|_{\mathrm{D}^k} \leq C(t)\left(\left|\mathrm{D}^k\right|\right)^{N+1}$$

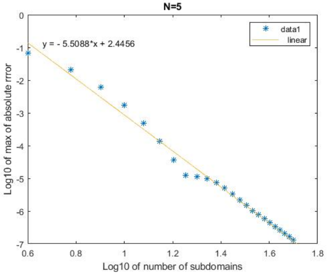

Figure: Error plot for N=5

For Further Reading I

Hesthaven, Jan S., Warburton, Tim.
Nodal Discontinuous Galerkin Methods.
Springer, 2008.

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Mode coupling mechanism for late-time Kerr tails.
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Summary

- We showed how Teukolsky equation can be used to get to the master coupled differential equation.
- We are applying this methodology to solve the Teukolsky equation.
  - Based on recent techniques developed by UMassD PhD student Ed McLain.
- Possible future works
  - Inclusion of other parameters like spin in the secondary BH, moving δ functions.
  - This will help in modeling the gravitational waves surrogately (without being computationally expensive)

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