A SUMMER PROJECT REPORT ON

STELLAR SPECTRA

Submitted by

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ACKNOWLEDGEMENTS

I would like to thank my project supervisor, Dr. Praveen Nagar, without whose awesome guidance I could not have learned anything I did learn in one month. He is very helping and patient. He explained the basics to me in a very simple and elegant way. I thank him for taking time out of his schedule and guiding me in this project.

I would thank my parents and family, who supported me enormously during my project. They encouraged me a lot and prevented me from getting defocused. I extend my sincere thanks to all of you.

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INTRODUCTION

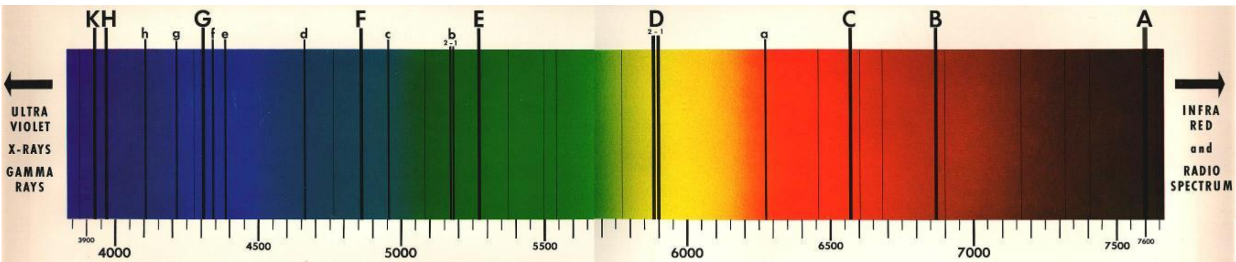
" A hundred years ago, Auguste Comte, a great philosopher, said that humans will never be able to visit the stars, that we will never know what stars are made out of, that that's the one thing that science will never ever understand, because they're so far away. And then, just a few years later, scientists took starlight, ran it through a prism, looked at the rainbow coming from the starlight, and said: “Hydrogen!” Just a few years after this very rational, very reasonable, very scientific prediction was made, that we'll never know what stars are made of. "
Michio Kaku

Stellar spectrum gives us a wealth of information about the star. Astronomers have discovered that the stars are made up of everyday elements found on the Earth. We can also deduce the stellar temperatures, radii, surface gravity, photospheric velocity fields, rotation rates, etc. just by using the electromagnetic spectrum obtained from the star.

In 1802, the English chemist William Wollaston noted that the spectrum of sunlight did not appear to be a continuous band of colors, but rather had a series of dark lines superimposed on it. Wollaston attributed the lines to natural boundaries between colours. Joseph Fraunhofer made a more careful set of observations of the solar spectrum in 1814 and found some 600 dark lines, and he specifically measured the wavelength of 324 of them. In 1864, Sir William Huggins matched some of these dark lines in spectra from other stars with terrestrial substances, demonstrating that stars are made of the same materials of everyday material rather than exotic substances. This paved the way for modern spectroscopy. But hold on. All this brouhaha about spectroscopy, spectrum. But what exactly is a spectrum? Stellar spectrum?

When light passes through a prism, it separates into the colors that make it up. White light changes to a swath of colors. This rainbow is called a spectrum. We can make spectra in many ways: with a prism, with drops of water (as in a real rainbow), or with gratings. Scientists build special instruments to separate light, usually with gratings. These instruments are called spectrographs.

When astronomers pass the light of a star through a spectrograph, they get a spectrum of the star. The spectrum looks like a regular rainbow of colors—except that there are dark lines in it. The position, the number and the size of the lines, give a lot of information about a star's atmosphere including its gaseous chemical content, its density and temperature. These lines are called spectral lines. Here is a spectrum of our sun:



The deep violet side on the left corresponds to a wavelength of about 3500 Angstroms and it ranges to the deep red on the right at about 7500 A, or roughly a span of 4000 A. There are dark vertical lines in the spectrum above. They are very narrow color bands in which there is less light, so they look dark. The light in the lines has been absorbed by the gases of the star's atmosphere, so they are called 'absorption' lines. Each element absorbs light of a particular frequency—a particular color. If that element is in the cool atmosphere of the star, those atoms will absorb the light at that color and produce the line. There are lots of lines in stellar spectra and they have different amounts of light missing or darkness.

Now that we are acquainted with the basic knowledge about a spectrum, let's dive into the details in the next few chapters. We'll explore the basic observational, computational and theoretical tools for measuring and interpreting stellar spectra and also study how spectral lines get modified due to photospheric velocity fields and stellar rotation.

BASIC PHYSICS

SPECTRAL CLASSIFICATION

Since even before the discovery of spectra, scientists had tried to find ways to categorize stars. By observing spectra, astronomers realized that large numbers of stars exhibit a small number of distinct patterns in their spectral lines. Classification by spectral features quickly proved to be a powerful tool for understanding stars.

The modern classification system is known as the Morgan–Keenan (MK) classification. Each star is assigned a spectral class from the older Harvard spectral classification and a luminosity class using Roman numerals, forming the star's spectral type. The classification is explained below.

Harvard classification system – The differences in spectra reflects different surface temperatures. Material on the surface of stars is "primitive". There is no significant chemical or nuclear processing of the gaseous outer envelope of a star once it has formed. Fusion at the core of the star results in fundamental compositional changes, but material does not generally mix between the visible surface of the star and its core.

Ordered from highest temperature to lowest, the seven main stellar types are O, B, A, F, G, K and M. O, B, and A type stars are often referred to as early spectral types, while cool stars (G, K, and M) are known as late type stars. The spectral characteristics of these types are summarized below:

Spectral Class Characteristics:

Class	Color	Temperature (K)
O	Blue	> 30,000
B	Blue-White	10,000-30,000
A	White	7,500-10,000
F	Yellow-White	6,000-7,500
G	Yellow	5,200-6,000
K	Orange - Yellow	3,700-5,200
M	Reddish-Orange	< 3,700

Within each of these seven broad categories, there are subclasses numbered 0 to 9. A star midway through the range between F0 and G0 would be an F5 type star. The Sun is a G2 type star.

Luminosity Classes – The Harvard scheme specifies only the surface temperature of the star. A more precise classification would also include the luminosity of the star. The standard scheme used for this is called the Yerkes classification (or MMK, based on the initials of the authors William W. Morgan, Philip C. Keenan, and Edith Kellman). This scheme measures the shape and nature of certain spectral lines to measure surface gravities of stars. The gravitational acceleration on the surface of a giant star is much lower than for a dwarf star (since $g = GM/R^2$ and the radius of a giant star is much larger than a dwarf). Given the lower gravity, gas pressures and densities are much lower in giant stars than in dwarfs. These differences manifest themselves in different spectral line shapes which can be measured.
The Yerkes scheme uses six luminosity classes:

Ia	Most luminous supergiants
Ib	Less luminous supergiants
II	Luminous giants
III	Normal giants
IV	Subgiants
V	Main sequence stars (dwarfs)

Thus the Sun would be more fully specified as a G2V type star.

MAGNITUDES AND COLOR INDICES

A basic observable quantity for a star is its brightness. Because stars can have a very broad range of brightness, astronomers commonly introduce a logarithmic scale called a magnitude scale to classify the brightness. Magnitude is given as –

$$m = -2.5\log\int_0^\infty F_\nu W(\nu)d\nu + const.$$

Where F_v is the flux of the star recorded in spectral interval specified by W(v). The constant is set according to the magnitude scale used (it is 0 for Vega).

One important distinction is between whether we are talking about the apparent brightness of an object, or its true brightness. The former is a convolution of the true brightness and the effect of distance on the observed brightness, because the intensity of light from a source decreases as the square of the distance (the inverse square law). Following that reasoning, there are two types of magnitudes : apparent magnitude and absolute magnitude.

Apparent Magnitude – The apparent magnitude of an object is the WYSIWYG magnitude. It is determined using the apparent brightness as observed, with no consideration given to how distance is influencing the observation. The apparent magnitude is easy to determine because we only need to measure the apparent brightness and convert it to a magnitude with no further thought given to the matter. However, the apparent magnitude is not so useful because it mixes up the intrinsic brightness of the star (which is related to its internal energy production) and the effect of distance (which has nothing to do with the intrinsic structure of the star).

Absolute Magnitude – Astronomers define the absolute magnitude to be the apparent magnitude that a star would have if it were hypothetically placed at a distance of 10 parsecs (which is 32.6 light years) from the Earth. We can do this if the true distance to the star is known because the inverse square law can be used to determine how its apparent brightness would change if it were moved from its true position to a standard distance of 10 parsecs. 10 parsecs is the distance astronomers have chosen for this standard. An upper-case "M" is used to denote an absolute magnitude.

A color index is defined by taking the difference in magnitudes at two different wavelengths. Using the U, B, and V color filters, there are three independent possible such differences. For example, the B-V color index is defined by taking the difference between the magnitudes in the blue and visual regions of the spectrum (technically it is a relative measure of the temperature of the star through the slope of Paschen continuum) and the U-B color index is the analogous difference between the UV and blue regions of the spectrum.

VELOCITY DISTRIBUTIONS

The velocities of the particles in a hot gas (such as in a star) follow the Maxwell-Boltzmann distribution. The fraction of particles in velocity interval (v,v+dv) is (in rectangular coordinates),

$$\frac{dN(v)}{N_{total}} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}} dv$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$ This is the Maxwellian speed distribution.

FOURIER TRANSFORMS

Fourier transforms are an indispensable tool for analyzing stellar spectra. Many calculations can be done with relative ease by switching over to the Fourier domain. The basic Fourier transform of a function F(x) is defined as

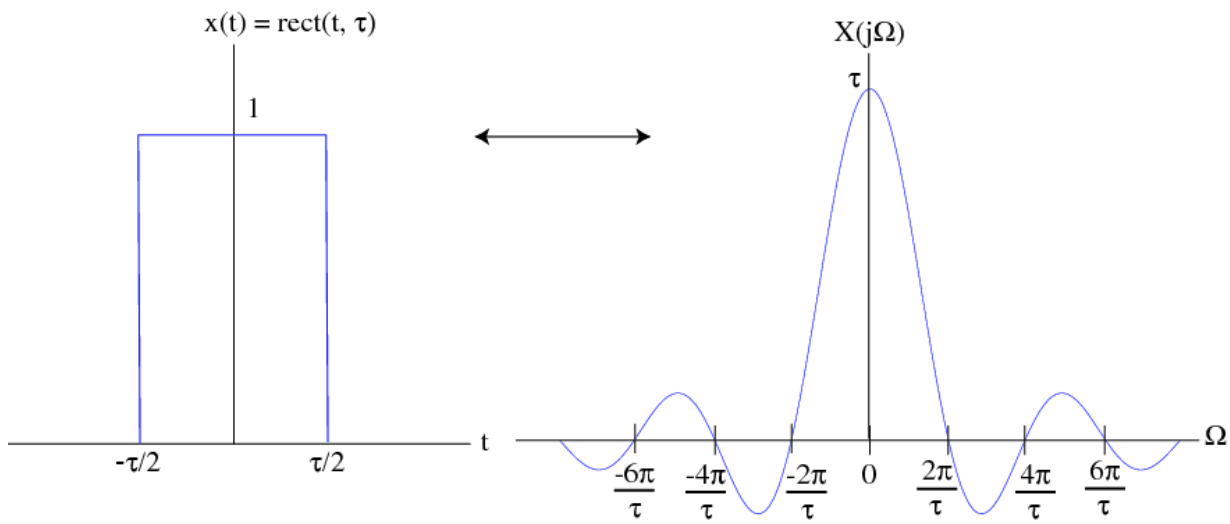
$$f(\sigma) = \int_{-\infty}^{\infty} F(x) e^{2\pi i x \sigma} dx$$

where x and σ are known as a Fourier pair. The inverse Fourier transform is

$$F(x) = \int_{-\infty}^{\infty} f(\sigma) e^{-2\pi i x \sigma} d\sigma$$

COMMON FOURIER TRANSFORMS

A box function transforms to a sinc function and vice versa.



A Gaussian transforms to another Gaussian.
Transform of a dispersion profile (known as Lorentzian function) –

$$F(x) = \frac{1}{\pi}\frac{\beta}{x^2+\beta^2} \xleftrightarrow{F.T.} f(\sigma) = e^{-2\pi\beta|\sigma|}$$

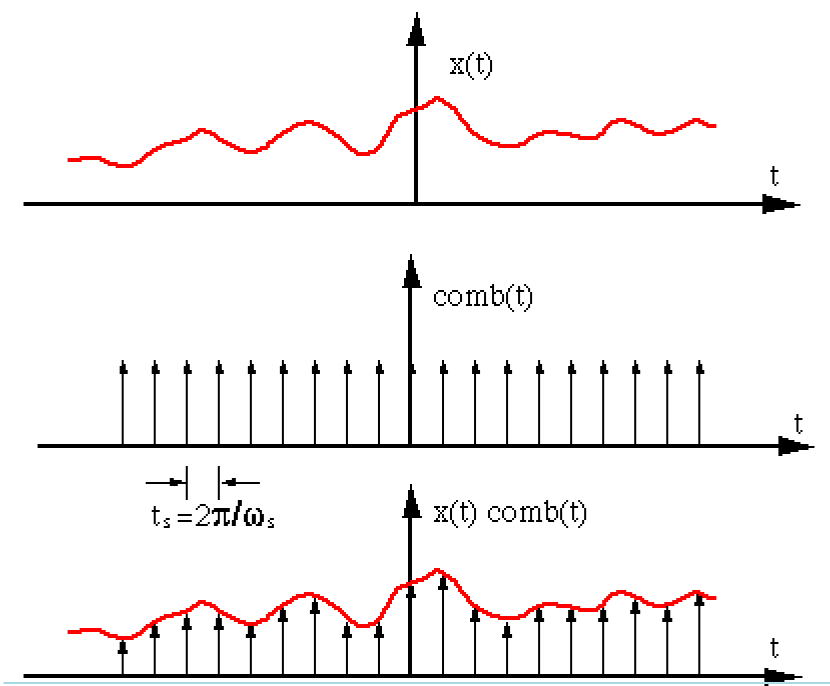
A delta function transforms to a constant and vice versa.

A Dirac comb (an infinite array of equally spaced δ-functions, also known as Shah function) transforms to another Dirac comb.

$$III(x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \xleftrightarrow{F.T.} III(\sigma) = \sum_{n=-\infty}^{\infty} \delta(\sigma - \frac{n}{\Delta x})$$

SAMPLING

The data we acquire is discrete and not continuous. Our data doesn't represent the true function. It can be written as a product of the "true" function x(t) and the Dirac comb III(t).

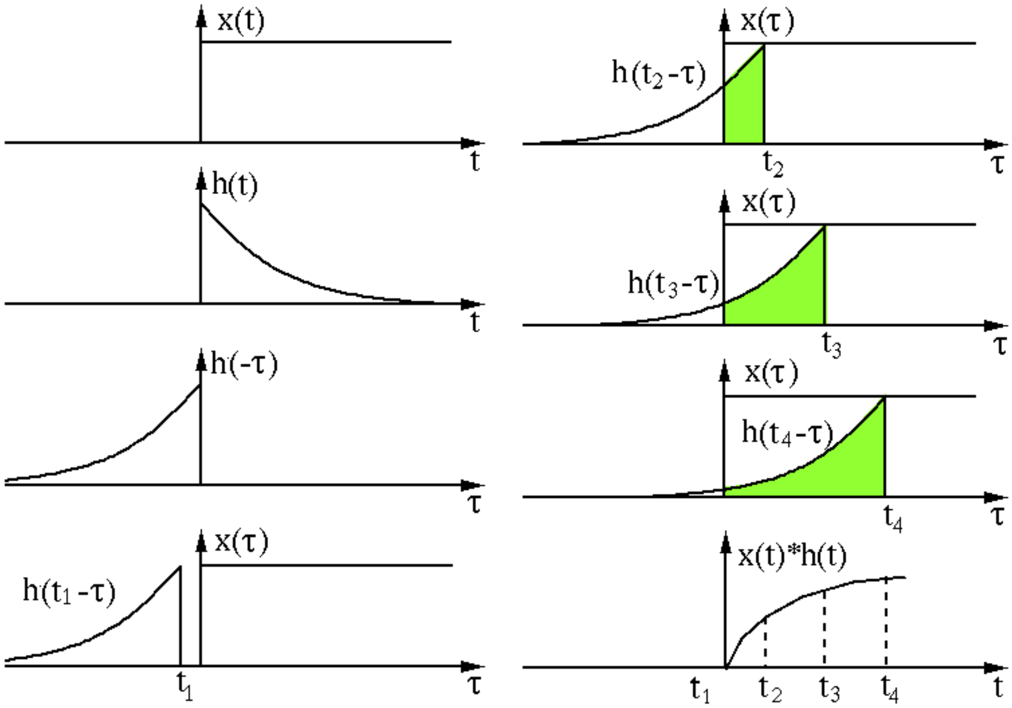


Also, we record data in an interval (finite time or finite bandwidth). $t_1 \rightarrow t_2$
An interval/window is expressed as multiplication with a box function, B(t) having a width equal to window limits of our observation.

D(t) = B(t)III(t)x(t)

CONVOLUTION

A convolution is an integral that gives as output the amount of overlap of one function as it is shifted over another function. It is useful because convolutions in one Fourier domain become simple multiplications in the other domain.



$$k(\sigma) = \int_{-\infty}^{\infty} f(\sigma_1) g(\sigma - \sigma_1) d\sigma_1 \quad \Rightarrow \quad k(\sigma) = f(\sigma) * g(\sigma) = g(\sigma) * f(\sigma)$$

Convolution with a δ-function gives the same function translated to the center of the δ-function.

$$F(x) * \delta(x - x_1) = F(x - x_1)$$

Convolution of two Gaussians gives a third Gaussian, convolution of two Lorentzian functions gives a third Lorentzian function, convolution of a Gaussian and a Lorentzian function gives a Voigt function.

USEFUL THEOREMS

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\|a\|} F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\|a\|} F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

ALIASING AND NYQUIST SAMPLING THEOREM

In many applications we need to create to a bridge between continuous-time signals ("analog") and discrete-time signals ("digital"). The Nyquist-Shannon sampling theorem establishes a sufficient condition for a sample rate that permits a discrete sequence of samples to capture all the information from a continuous time signal of finite bandwidth. Reconstruction of analog from digital signals is done via the Whittaker-Shannon interpolation formula. The theorem states that "If a function x(t) contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced (1/2B) seconds apart."

The sampling rate of 2B is called the Nyquist frequency. If the Nyquist theorem is not satisfied during sampling then aliasing results, i.e., the transform has overlapping regions.

Say,

D(x) = B(x) III(x) F(x)

Its Fourier transform,

D(σ)=b(σ)*III(σ)*f(σ)

Assume that B(x) is so wide, that b(σ) is essentially an impulse.

=> d(σ) = III(σ) * f(σ)

(since convolution with δ(σ), the impulse, gives the same function back)

If the data spacing in D(x) is Δx, then the data spacing in d(σ) is 1/Δx
If f(σ) reduces to zero for σ<0.5/ Δx then there will be no aliasing. Otherwise there will be some overlapping.

$\sigma_N = 0.5/\Delta x$ is known as the Nyquist frequency.

NUMERICAL CALCULATION OF TRANFORMS (FFT)

The Fast Fourier Transform method is discussed. Assumptions/Requirements are

1) Number of points in the function to be transformed should be integral powers of 2 (if its not the case, then it can be done by extension of the data array by adding zeros.)
2) All data points must be equally spaced along the abscissa (x-axis)

$$D(x) = D(j\Delta x) = D(j)$$

where j is an integer index from 0 to N-1 (N is the number of points). Δx is the step.

Now the transform is,

$$d(\sigma) = \sum_{x_j=-\infty}^{\infty} D(x_j) e^{2\pi i x_j \sigma} \Delta x = \sum_{0}^{N-1} D(j) e^{2\pi i (j\Delta x)\sigma} \Delta x$$

Where d(σ) is written in the Fourier series form.

Also $\sigma = k\Delta\sigma$ where $\Delta\sigma$ is the spacing of points in d(σ) and k is an integer.

$$d(\sigma) = d(k) = \sum_{j=0}^{N-1} D(j) e^{2\pi ijk\Delta x\Delta\sigma} \Delta x$$

The highest frequency at which we can obtain information about the transform is the Nyquist frequency $\sigma_N = 1/(2\Delta x)$

If there are N points for D(x), then there are N values for d(σ). Also, d(σ) extends equally far in negative as in positive σ, i.e., σ_N would be N/2 points away from zero.

$$\Delta\sigma = \frac{\sigma_N}{N/2} = \frac{1}{\Delta x N} \Rightarrow \Delta\sigma\Delta x = \frac{1}{N}$$

Hence,

$$d(k) = \sum_{j=0}^{N-1} D(j) e^{2\pi ijk/N} \Delta x$$

This summation is evaluated for FFT assuming Δx=1.

The inverse transform would be,

$$D(j) = \sum_{k=0}^{N-1} d(k) e^{-2\pi ijk/N} \Delta\sigma$$

But ΔσΔx=1/N, hence

$$D(j) = \frac{1}{\Delta x N} \sum_{k=0}^{N-1} d(k) e^{-2\pi ijk/N}$$

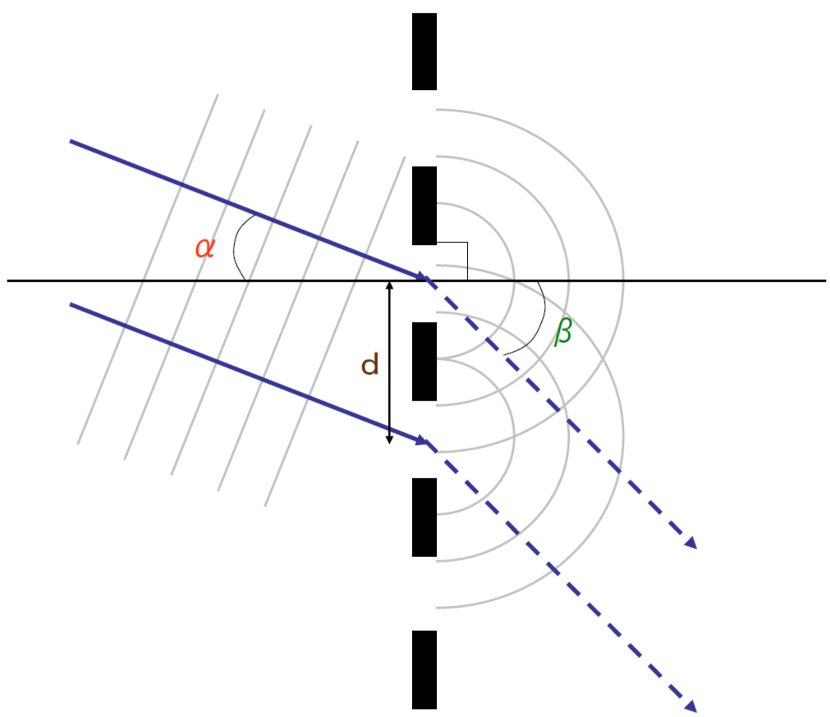
GRATINGS

The basic setup used in any spectrograph uses a diffraction grating. Diffraction grating is the element which disperses light into its constituents wavelengths. The incidence of a wave on the grating and the resultant spectrum formation is discussed below.

DIFFRACTION GRATING

An incident plane scalar wave

$$F(x,t) = F_0(t)\exp\left(\frac{2\pi i x \sin\alpha}{\lambda}\right)$$



The grating transmission is described by the function G(x)

$$G(x) = \begin{cases} 1 & for\, x\, values\, corresponding\, to\, slits \\ 0 & for\ all\ other\ values\ of\ x \end{cases}$$

The wave that emerges from the rear side of the grating is the product of F(x,t) and G(x). The resultant wave in arbitrary directions (say, β) will be sum of the contributions, with proper phase shift from along x-coordinate (basically Huygens' principle).

$$g(\beta) = \int_{-\infty}^{\infty} F(x,t)G(x)e^{2\pi ix\sin\beta/\lambda}dx$$

Where 2π(x sinβ)/λ is the required phase difference along the grating.

$$g(\beta) = F_0 \int_{-\infty}^{\infty} G(x)e^{2\pi i(x\sin\alpha + x\sin\beta)/\lambda}dx$$

where we substituted the original F(x,t).

We choose to measure x in units of λ. Also sinα+sinβ=θ

$$g(\theta) = F_0 \int_{-\infty}^{\infty} G(x)e^{2\pi ix\theta}dx$$

This tells us a profound thing. The resultant waves behind the grating can be written as a Fourier transform of grating transmission function G(x).

$$G(x) = B_1(x)*III(x)B_2(x)$$

G(x) is taken to be purely real, i.e., an amplitude function. A more general G(x) would be complex. A purely imaginary G(x) will be a phase grating. The box $B_1(x)$ represents transmission through a single slit, and so has a width 'b'. Box $B_2(x)$ matches total width of the grating, 'W'. The spacing between the δ-function in III(x) is 'd'.

Different orders of maxima 'n', arise from multiple slits and 'b' (single slit width) sets the diffraction envelope, 'W' sets the width of individual interference maxima.

$$g(\theta) = F.T.\{G(x)\} = III(\theta) * \left(\frac{W \sin \pi\theta W}{\pi\theta W} \right) \frac{b \sin \pi\theta b}{\pi\theta b}$$

Now, $III(\theta) = \sum_n \delta(\theta - \frac{n}{d})$

$$g(\theta) = \sum_n \frac{W \sin \pi(\theta - n/d)W}{\pi(\theta - n/d)W} \frac{b \sin \pi\theta b}{\pi\theta b}$$

Square of this wave amplitude function is proportional to the light intensity. The distribution of light in diffraction pattern of grating is proportional to $g^2(\theta)$. There is maxima in g(θ) when θ-n/d=0. Condition for maxima is,

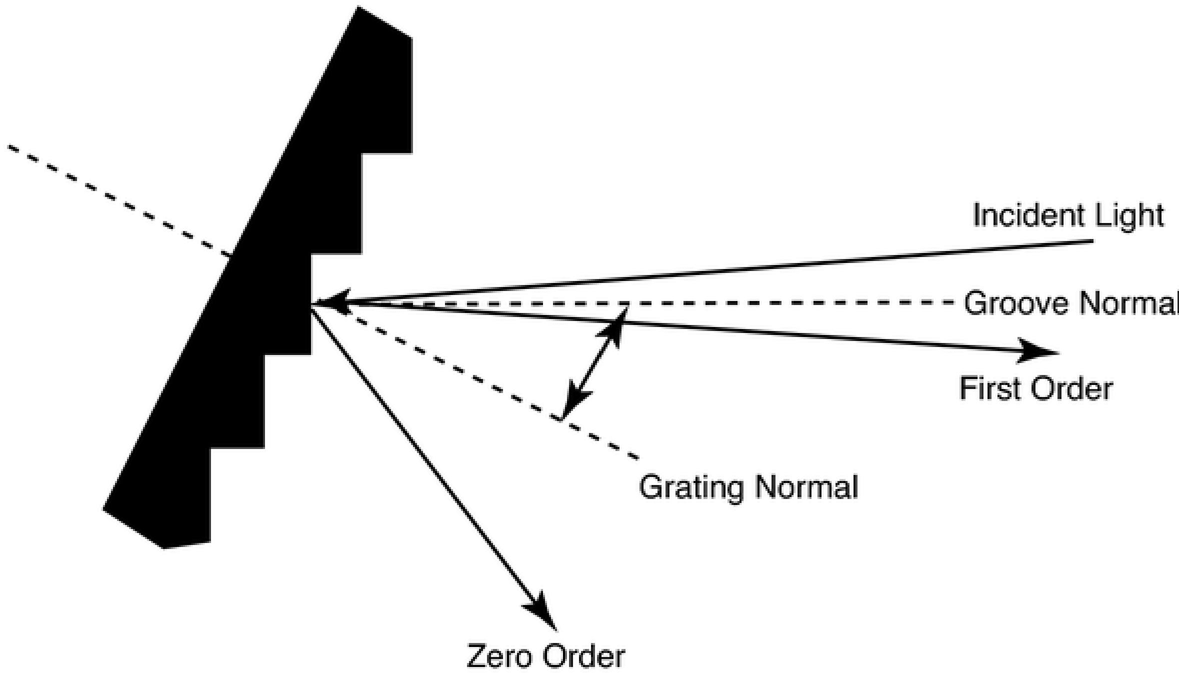
$$\frac{n\lambda}{d} = \theta = \sin\alpha + \sin\beta$$

('d' was measured in terms of wavelength, now it is measured in the standard units). This equation is also known as the grating equation.

BLAZED REFLECTION GRATINGS

In normal gratings, the maxima at θ=0 is the brightest, the most amount of light is lost there. Sometimes, we may need better intensity for higher orders. For doing this, we shift the diffraction envelope relative to the interference pattern to those θ values in which we wish to work. Fourier shift theorem tells us we can accomplish this by introducing a phase term in $B_1(x)$. Such a grating is known as a blazed grating.

We can achieve a phase shift by introducing prisms in the slits. But that results in intensity loss. So we use a reflection grating. The slit width and slit spacing are the same in this case, i.e., there is no slit. The light incident on the grating reflects and forms the resultant spectrum.



Now, b=d and the facet angle (angle between grating normal and groove normal) is φ. The equation for transmission grating holds for reflection gratings as well.

The diffraction envelope of g(θ) is given by,

$$\frac{b\sin\pi\theta b}{\pi\theta b} = \frac{b\sin\left[\pi b(\sin\alpha + \sin\beta)\right]}{\left[\pi b(\sin\alpha + \sin\beta)\right]}$$

Tilting the reflection facet through φ changes phase of reflected wave, i.e., α→α – φ and β→β – φ . Then the envelope light distribution is proportional to,

$$I(\beta) = \left\{\frac{\sin\left[(\pi b/\lambda)(\sin(\alpha-\varphi)+\sin(\beta-\varphi))\right]}{\left[(\pi b/\lambda)(\sin(\alpha-\varphi)+\sin(\beta-\varphi))\right]}\right\}^2$$

Or after using the grating equation,

$$I(\beta) = \left\{\frac{\sin\left[\{n\pi b/d\}\{\cos\varphi - [\sin\varphi/\tan((\alpha+\beta)/2)]\}\right]}{\left[\{n\pi b/d\}\{\cos\varphi - [\sin\varphi/\tan((\alpha+\beta)/2)]\}\right]}\right\}^2$$

DETECTORS

The spectrum is obtained via the detectors. There are many kinds of detectors: Photographic plates, Photomultiplier tubes (PMT), Charge-coupled devices (CCD). These days mostly CCDs are used.

Photographic plates are basically crystals of silver halides (AgCl, AgI, usually AgBr). It is suspended in thin gelatin layer (15-30μm thick) called emulsion. The gelatin layer is supported by a substrate of glass or celluloid.
Photomultiplier tubes (PMT) are based on the principle of photoelectric effect. These detectors multiply the current produced by incident light on the photocathode which gets multiplied through the dynodes, reaches the anode and produces a signal.

QUANTUM EFFICIENCY AND SPECTRAL RESPONSE

Quantum efficiency is the ratio of the number of photons collected by the device to the number of photons incident on the device.

$$q(\lambda) \equiv \frac{No.\ of\ \ photons\ detected}{No.\ of\ \ photons\ incident}$$

Spectral response is the relative efficiency of a detector with wavelength, e.g., q(λ), for photographic plate, exposure (product of level of illumination(watts per square meter) times the exposure time). It is proportional to total number of photons impressed upon a unit area of the plate. Spectral response of the photographic plate is reciprocal of exposure needed to give a specified level of darkening on the plate.
For photomultiplier tubes, we list q(λ) or photocathode current per watt of incident illumination. Suppose incident radiation has strength of P watts and consists of N photons/sec. => P = Nhc/λ

If current generated by cathode is $i = q(\lambda)Ne$, then,

$$\frac{i}{P} = \frac{\lambda e}{hc} q(\lambda) \Rightarrow \frac{i}{P} = 8.07 * 10^{-5} \lambda q(\lambda) \qquad (for\, \lambda\ \ in\ angstroms)$$

DETECTOR DARK OUTPUT

Detector dark output refers to the dark current, the current when illumination is zero. Photographic plates show background 'fog' due to cosmic rays and radioactive trace elements in vicinity of the plate, most of it due to thermal excitation of silver halides in emulsion. Hence, we should refrigerate spectroscopic plates until used. Generally they are kept in liquid nitrogen. A discriminator can be used to cut out small noise pulses.

LINEARITY OF PMT

When output of detector is proportional to amount of incident radiation, then the detector is linear. PMTs are generally linear. There are a number of ways it can be destroyed –

1)Photomultiplier fatigue (usage of PMT for a long duration)
2) Rate of pulses greater than counting speed of electronic circuitry
3) Effect of Earth's magnetic field
4) Gain stability (the gain may not be stable)

The **spatial resolution** of the detector is also important. The instrumental profile (or δ-function response) of the detector should be measured and removed from the observations. The Fourier transform of the instrumental profile is known as the Modulation Transfer Function (MTF).

NOISE

We need to remove the noise from the signal. There is photon noise from starlight, in sky background, also there is equipment noise. The fluctuations in photon rate are described by Bose-Einstein statistics, for a thermal source of characteristic temperature T, mean square fluctuation Δn photons/sec, is

$$\Delta n = n^{1/2}\left(1+\frac{1}{e^{h\nu/kT}-1}\right)^{1/2}$$

Since hv>>kT generally $\Delta n = n^{1/2}$ (Poisson statistics). Equipment noise also follows Poisson statistics.

We can combine main sources of noise as follows. Define the rate of photon counts for the light source to be L per second and rate of counts corresponding to equipment noise and other background to be B. Number of recorded counts in an integration time t is n = (L+B)t
Our signal is comprised of number of light counts N = Lt
It is found by subtracting background from total

$$N = n - Bt = n - b \qquad \text{where } b = Bt$$

$$\Delta N = [\Delta n^2 + \Delta b^2]^{1/2}$$

If deviations are sufficiently well represented by a Gaussian distribution,

$$\Delta n = n^{1/2} \quad \text{and} \quad \Delta b = b^{1/2}$$

$$\Rightarrow \varepsilon \equiv \frac{\Delta N}{N} = \frac{(n+b)^{1/2}}{N}$$

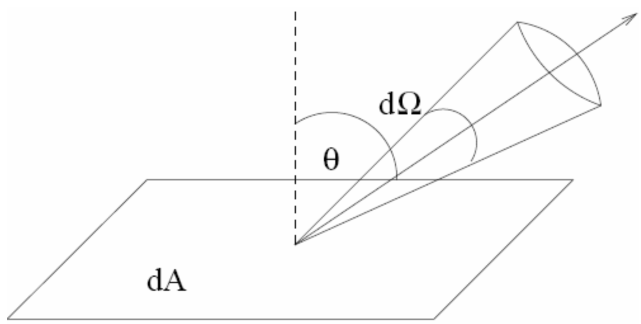
$$\Rightarrow \varepsilon \equiv \frac{[(L+B)t + Bt]^{1/2}}{Lt} = \left(\frac{1 + 2B/L}{Lt}\right)^{1/2}$$

if B/L << 1 => $\varepsilon = (Lt)^{-1/2} = N^{-1/2}$ => pure signal photon noise.

if B/L >> 1 => $\varepsilon = [(2B/L)/Lt]^{1/2}$ Since B/L>>1 hence ε is large unless t is made very long.

BASIC RADIATION TERMINOLOGY

SPECIFIC INTENSITY



Suppose we have radiation from a spherical star. Take ΔA, a small area of the spherical surface. Then the specific intensity I_ν is defined as,

$$I_\nu \equiv \lim \frac{\Delta E_\nu}{\cos\theta\ \Delta A\ \Delta\omega\ \Delta t\ \Delta\nu} = \frac{dE_\nu}{\cos\theta\ dA\, d\omega\ dt\ d\nu}$$

where θ is the angle from the normal to the line of sight and dω is denoted as dΩ in the figure (differential solid angle).

Mean intensity is defined as directional average of specific intensity.

$$\overline{I_\nu} \equiv \frac{1}{4\pi}\oint I_\nu d\omega$$

FLUX

Flux is a measure of net energy flow across an area ΔA, in time Δt, in a spectral range Δν,

$$F_\nu \equiv \lim \frac{\sum \Delta E_\nu}{\Delta A\ \Delta t\ \Delta\nu} = \frac{\oint dE_\nu}{dA\, dt\ d\nu}$$

$$\Rightarrow F_\nu = \oint I_\nu \cos\theta \, d\omega$$

If we look at a point on the physical boundary of a radiating sphere,

$$F_\nu = \int_0^{2\pi} d\phi \int_0^{\pi} I_\nu \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} I_\nu \sin\theta \cos\theta \, d\theta + \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} I_\nu \sin\theta \cos\theta \, d\theta$$

i.e., flux leaving surface plus flux entering surface. In case at hand (star) second term is zero. If there is no azimuthal dependence for I_ν,

$$F_\nu = 2\pi \int_0^{\pi/2} I_\nu \sin\theta \cos\theta \, d\theta$$

if I_ν is independent of direction (no θ dependence),

$$\Rightarrow F_\nu = \pi I_\nu$$

I_ν is independent of distance from source whereas F_ν obeys standard inverse square law. I_ν can only be measured directly if we can resolve the radiating surface, otherwise we necessarily measure F_ν.

THE K-INTEGRAL AND RADIATION PRESSURE

Mean intensity is $\overline{I_\nu} \equiv \frac{1}{4\pi} \oint I_\nu d\omega$ and flux $\Rightarrow F_\nu = \oint I_\nu \cos\theta \, d\omega$

Also, $K_\nu = \frac{1}{4\pi} \oint I_\nu \cos^2\theta \, d\omega$ which is related to radiation pressure.

Say we have photons in a box. Calculate the pressure applied by the photons on the box walls. Photons have momentum. They transfer momentum to the solid wall in a manner analogous to kinetic gas theory. The component of momentum normal to the wall taken per unit time and area is the pressure,

$$dP_\nu = \frac{1}{c}\frac{dE_\nu \cos\theta}{dt\, dA}$$

where θ is the angle the photon trajectory forms with the normal to the wall. In terms of specific intensity,

$$dP_\nu = P_\nu \, d\nu \, d\omega = \frac{I_\nu}{c}\cos^2\theta \, d\nu \, d\omega$$

Integrated over solid angle,

$$P_\nu d\nu = \oint \frac{I_\nu}{c}\cos^2\theta \, d\nu \, d\omega$$

"Monochromatic" pressure refers to pressure exerted by photons with frequency in range dv.

$$\Rightarrow P_\nu = \frac{4\pi K_\nu}{c}$$

Special case – assume I_ν is independent of direction, i.e., factor I_ν out of the integral,

$$\Rightarrow P_\nu = \frac{4\pi}{3c} I_\nu$$

and total radiation pressure is,

$$P_R = \frac{4\pi}{3c}\int_0^\infty I_\nu \, d\nu$$

Also, $\pi \int_0^\infty I_\nu \, d\nu = \sigma T^4$ (Stefan-Boltzmann law, $\int_0^\infty F_\nu \, d\nu = \sigma T^4$ and $F_\nu = \pi I_\nu$)

$$\Rightarrow P_R = \frac{4\sigma}{3c} T^4$$

where T is the kinetic gas temperature.

THE ABSORPTION COEFFICIENT & OPTICAL DEPTH

Consider radiation in specified direction shining on a small thickness of material which itself is so cool that it doesn't radiate measurably. Intensity of light is found experimentally to be diminished upon passage through the layer by an amount dI_ν where

$$dI_\nu = -\kappa_\nu \rho I_\nu \, dx$$

where κ_ν is the absorption coefficient (cm^2/g) or mass absorption coefficient, ρ is density in g/cc and dx is in cm.

Two processes contribute to κ_ν –

1) True absorption – photon is destroyed and energy thermalized.
2) Scattering – photon is deviated in direction and removed from solid angle being considered.

κ_ν is really an extinction coefficient.

$$\tau_\nu = \int_0^L \kappa_\nu \rho \, dx$$

where τ_ν is optical depth and x is the geometrical depth.

$$\Rightarrow dI_\nu = -I_\nu \, d\tau_\nu$$

$$\Rightarrow I_\nu = I_\nu^0 e^{-\tau_\nu}$$

This is the usual simple extinction law.

THE EMISSION COEFFICIENT & SOURCE FUNCTION

Increment of radiation emitted in a specific direction is

$$dI_\nu = j_\nu \rho \, dx$$

where j_ν is the emission coefficient (ergs per second per rad^2 per c per second per gram)

Physical processes contributing to j_ν –

1) Real emission – creation of a photon
2) Scattering of photon into the direction being considered. Not diffraction but absorption followed immediately by a reemission of photon from same atomic transition.

Ratio of emission to absorption has the same units as I_ν

Source function $$S_\nu \equiv \frac{j_\nu}{\kappa_\nu}$$

PURE ISOTROPIC SCATTERING

Pure isotropic scattering – All emitted energy is due to photons being scattered into direction under consideration. Contribution dj_ν to emission from solid angle $d\omega$ is proportional to $d\omega$ and to absorbed energy $\kappa_\nu I_\nu$. It is isotropically reradiated so the fraction per unit solid angle is $1/4\pi$.

$$dj_\nu = \frac{1}{4\pi}\kappa_\nu I_\nu d\omega$$

$$\Rightarrow j_\nu = \frac{1}{4\pi}\oint \kappa_\nu I_\nu d\omega$$

But κ_ν is generally ω independent,

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{\oint I_\nu d\omega}{4\pi}$$

$$\Rightarrow S_\nu = \overline{I_\nu}$$

i.e., source function for pure isotropic scattering is mean intensity.

PURE ABSORPTION

When all absorbed photons are destroyed and all emitted photons are created with a distribution governed by physical state of material => that is pure absorption. Common usage of term has narrowed its meaning to that physical state of material called thermodynamic equilibrium.

Emission from gas in thermodynamic equilibrium is given by laws of black body radiator. Source function given by Planck's Radiation Law,

$$S_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

Combine scattering with pure absorption. Add photons contributed from two mechanisms.

=> emission coefficient is $j_\nu = \kappa_\nu^S \overline{I_\nu} + \kappa_\nu^A B_\nu(T)$

Similarly add absorption coefficients to obtain total absorption coefficient. Source function for this combination is then ratio of total emission coefficient to total absorption coefficient,

$$S_\nu = \frac{\kappa_\nu^S}{\kappa_\nu^S + \kappa_\nu^A}\overline{I_\nu} + \frac{\kappa_\nu^A}{\kappa_\nu^S + \kappa_\nu^A}B_\nu(T)$$

EINSTEIN COEFFICIENTS

When dealing with spectral lines or bound-bound transitions, we can describe the spontaneous probabilities for emission in terms of atomic constants. Consider spontaneous transition between upper level, u, and lower level, l, separated by energy hv. Assume emission is isotropic. Probability that atom will emit its quantum of energy in time dt and in a solid angle dω is A_{ul} dt dω. A_{ul} is the Einstein probability coefficient for spontaneous emission.

If there are N_u excited atoms per unit volume, the contribution of spontaneous emission to emission coefficient is, $j_\nu \rho = N_u A_{ul} h\nu$

Probability for stimulated emission giving a quantum in time interval dt in solid angle dω is, $B_{ul} I_v$ dt dω , where directional dependence of I_v becomes important since it fixes directional dependence of these new photons. B_{ul} is the Einstein probability coefficient for stimulated emission.

True absorption probability is defined in the same way and the proportionality constant is denoted by B_{lu} . Mass absorption coefficient for this bound-bound transition can be expressed in terms of B's considering radiation absorbed per unit path length from intensity beam I_v ,

$$\kappa_\nu \rho I_\nu = N_l \left(B_{lu} I_\nu\right) h\nu - N_u \left(B_{ul} I_\nu\right) h\nu$$

where N_l is the population of lower level per unit volume.

$$\Rightarrow \kappa_\nu \rho = N_l B_{lu} h\nu - N_u B_{ul} h\nu$$

BLACK BODY RADIATION

EMPIRICAL RELATIONS

Blackbody spectrum is continuous, isotropic and unpolarized. Intensity of continuum is found to depend only on frequency and temperature of blackbody. Two laws are found empirically.

First of these is a scaling relation,

$$I_\nu = \nu^3 F\left(\frac{\nu}{T}\right) \quad or \quad I_\lambda = \frac{c^4}{\lambda^5} F\left(\frac{c}{\lambda T}\right)$$

Where F(.) is a unique function that can be tabulated from measurements.

Now, say u=v/T => $I_\nu = T^3 u^3 F(u)$

From observations,

$$\lambda_{\max} T = 0.28978 \ \ cm\, K$$

Wien's law

Second law is the Stefan-Boltzmann law, i.e., total power output is specified purely by temperature according to

$$\int_0^\infty F_\nu d\nu = \sigma T^4$$

Where F_ν is the flux and σ is the Stefan-Boltzmann constant.
σ = 5.6703*10^{-5} erg/sec cm^2 deg^4

It follows from Wien's law. Let u=v/T

$$I_\nu = T^3 u^3 F(u)$$

$$\Rightarrow \int_0^\infty I_\nu d\nu = \int_0^\infty u^3 T^3 F(u) T\, du = T^4 \int_0^\infty u^3 F(u)\, du = const.\, T^4$$

At low frequencies, $I_\nu = \frac{2kT\nu^2}{c^2}$, which is the Rayleigh-Jean's approximation.

At high frequencies, $I_\nu = const.\, \nu^3 e^{-const\,\nu/T}$, which is the Wien's approximation.

PLANCK'S RADIATION LAW

We derive the Planck's law using the two-level atom. Upper level population N_u, lower level population is N_l and since atoms are in container in thermodynamic equilibrium => their population are related by,

$$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-h\nu/kT}$$

where energy difference $X_u - X_l = hv$

Equilibrium condition implies that all the ways for electron to go from u to l must be balanced by the return paths.
Number of spontaneous emissions per second per unit solid angle per unit volume is $N_u A_{ul}$. Rate of stimulated emission per unit solid angle and volume is $N_u B_{ul} I_v$ where I_v is taken at v appropriate to the transition. Absorption pumps electron upward, so for this rate we write $N_l B_{lu} I_v$

$$N_u A_{ul} + N_u B_{ul} I_\nu = N_l B_{lu} I_\nu$$

$$\Rightarrow I_\nu = \frac{A_{ul}}{B_{lu}\frac{N_l}{N_u} - B_{ul}}$$ (Ratio of emission to absorption)

this has the form of source function.

Substitute for population ratio according to the excitation equation,

$$I_\nu = \frac{A_{ul}}{\frac{g_l}{g_u} B_{lu} e^{h\nu/kT} - B_{ul}}$$

This must go to Rayleigh-Jean's approximation in limit of small v. Expand exponential keeping only 1st order terms,

$$I_\nu = \frac{A_{ul}}{\frac{g_l}{g_u} B_{lu} - B_{ul} + \frac{g_l}{g_u} B_{lu} \frac{h\nu}{kT}} \quad for \frac{h\nu}{kT} << 1$$

which must be made equal to $2kT\nu^2/c^2$. This can be done only if,

$$B_{ul} = \frac{B_{lu} g_l}{g_u} \quad \& \quad A_{ul} = \frac{2h\nu^3}{c^2} B_{ul}$$

Substitute into $I_\nu = \frac{A_{ul}}{B_{lu} \frac{N_l}{N_u} - B_{ul}}$,

$$\Rightarrow I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$ which is the Planck's radiation law.

This is also $B_\nu(T)$, blackbody source function. From this equation, we can derive Wien's law and Stefan-Boltzmann law.

RADIATIVE AND CONVECTIVE ENERGY TRANSPORT

THE TRANSFER EQUATION AND ITS FORMAL SOLUTION

The major mode of transport of energy is radiation. Consider radiation traveling in direction s. The change in specific intensity, dI_ν, over an increment of path length, ds, is sum of losses and gains,

$$dI_\nu = -\kappa_\nu \rho I_\nu ds + j_\nu \rho ds$$

Divide by $\kappa_\nu \rho ds = d\tau_\nu$,

$$\Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{\kappa_\nu}$$

$$\Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$ This is the equation of radiative transfer.

Integrate using standard integrating factor scheme.

$$I_\nu(\tau_\nu) = f\, e^{b\tau_\nu} \qquad (Say)$$

' f ' is the function to be determined.

$$\Rightarrow \frac{dI_\nu}{d\tau_\nu} = f\, b\, e^{b\tau_\nu} + e^{b\tau_\nu} \frac{df}{d\tau_\nu}$$

$$\Rightarrow bI_\nu + e^{b\tau_\nu} \frac{df}{d\tau_\nu} = -I_\nu + S_\nu$$

=> b = -1 and $e^{-\tau_\nu} \dfrac{df}{d\tau_\nu} = S_\nu$

$$\Rightarrow f = \int_0^{\tau_\nu} S_\nu(t_\nu) e^{t_\nu} \, dt_\nu + c_0$$

$$\Rightarrow I_\nu(\tau_\nu) = e^{-\tau_\nu} \int_0^{\tau_\nu} S_\nu(t_\nu) e^{t_\nu} \, dt_\nu + c_0 e^{-\tau_\nu}$$

Integration constant can be written as $c_0 = I_\nu(0)$

$$\Rightarrow I_\nu(\tau_\nu) = \int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(\tau_\nu - t_\nu)} dt_\nu + I_\nu(0) e^{-\tau_\nu}$$

This is the transfer equation. I_ν is a function of optical depth along a line. This equation can be applied directly in studies of interstellar medium since observation at a point on celestial sphere corresponds to observation along such a line. For stellar atmospheres, it is conventional to define optical depth relative to star along a stellar radius and not along the line of sight. Appropriate projection factor must then be incorporated.

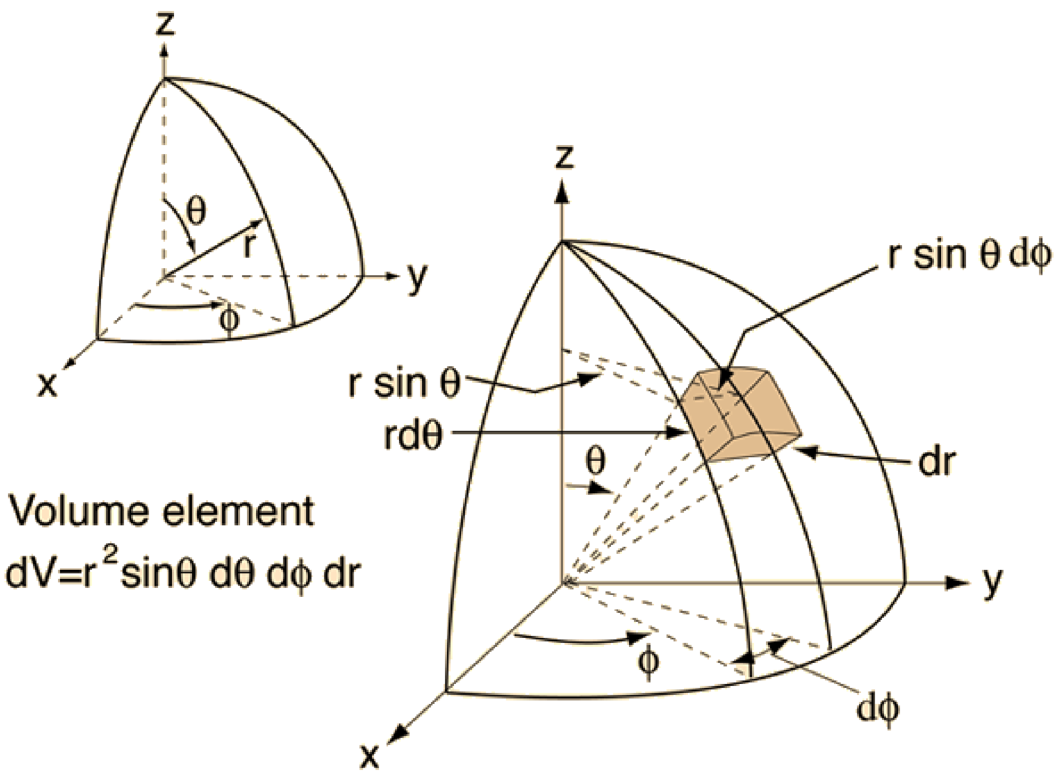
THE TRANSFER EQUATION FOR DIFFERENT GEOMETRIES

Assume spherical stars => spherical coordinates, the z axis is towards observer. Because we are concerned with geometry, the transfer equation can be written in the form,

$$\frac{dI_\nu}{\kappa_\nu \rho \, dz} = -I_\nu + S_\nu$$

In general,

$$\frac{dI_\nu}{dz} = \frac{\partial I_\nu}{\partial r} \frac{dr}{dz} + \frac{\partial I_\nu}{\partial \theta} \frac{d\theta}{dz}$$



From figure,

$$dr = \cos\theta \, dz$$

$$rd\theta = -\sin\theta \, dz$$

Hence transfer equation,

$$\Rightarrow \frac{\partial I_\nu}{\partial r}\frac{\cos\theta}{\kappa_\nu \rho} - \frac{\partial I_\nu}{\partial \theta}\frac{\sin\theta}{\kappa_\nu \rho r} = -I_\nu + S_\nu$$

This form of the equation is used in stellar interiors and in calculation of very thick stellar atmospheres such as those found in supergiants and possible giants. Generally, in most stars the geometrical thickness of photosphere is very small compared to stellar radius. Solar photosphere is ~700km thick or ~0.1% of solar radius. => Plane parallel approximation can be made => θ doesn't depend on z => no second term in above equation,

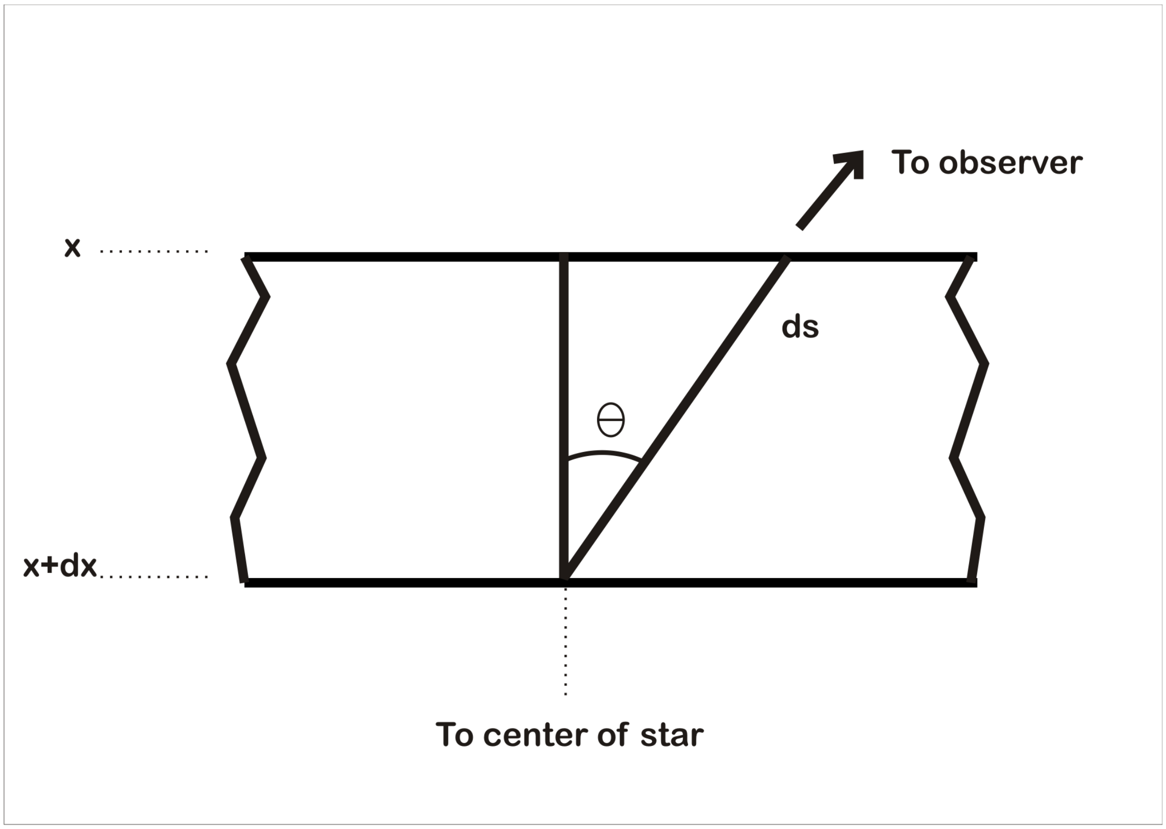
$$\Rightarrow \cos\theta \frac{dI_\nu}{\kappa_\nu \rho dr} = -I_\nu + S_\nu$$

It is a custom/convention to adopt a new geometrical depth variable, x, for plane geometry case defined by dx=-dr

$$\Rightarrow \cos\theta \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \qquad (\because d\tau_\nu = \kappa_\nu \rho dx)$$

This is the basic form of transfer equation used in the central arena of stellar atmospheres.

Optical depth defined in this way is measured along x and not along the line of sight which is at some angle, θ, as shown.



$$\Rightarrow ds = dx \sec\theta$$

This amounts to replacing τ_ν with $-\tau_\nu \sec\theta$. The negative sign comes from choosing dx = - dr.

$$\Rightarrow I_\nu(\tau_\nu) = -\int_c^{\tau_\nu} S_\nu(t_\nu) e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$

The integration limit, c, replaces the $I_\nu(0)$ integration constant. This is done because the boundary conditions are clearly different for radiation having $\theta \geq 90°$ (radiation inward) than for that having $\theta \leq 90°$ (radiation outward).

In the first case $I_\nu(0)=0$ for a normal isolated star, where $\tau_\nu = 0$ is taken to be outer boundary of atmosphere. Radiation from other stars, galaxies is completely negligible compared to star's own radiation. => For inward directed radiation,

$$I_\nu^{in}(\tau_\nu) = -\int_0^{\tau_\nu} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$

For second case, we consider radiation at depth τ_ν and deeper until no more radiation can be seen at our station. In other words, the integration limit is $\tau_\nu = \infty$ => For outward directed radiation

$$I_\nu^{out}(\tau_\nu) = \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$

=> Full intensity at position τ_ν is then

$$I_\nu(\tau_\nu) = I_\nu^{out}(\tau_\nu) + I_\nu^{in}(\tau_\nu)$$

$$= \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu - \int_0^{\tau_\nu} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$

One must require $\underset{\tau_\nu \to \infty}{Lt} \; S_\nu e^{-\tau_\nu} = 0$ to ensure that the integral exists. Real stars meet this condition effortlessly.

Special case → at stellar surface

$$I_\nu^{in}(0) = 0$$

$$I_\nu^{out}(0) = \int_0^\infty S_\nu e^{-t_\nu \sec\theta} \sec\theta \, dt_\nu$$

The last equation is relevant to solar work. For most stars, we must deal with the flux.

THE FLUX INTEGRAL

Expand $F_\nu = \oint I_\nu \cos\theta \, d\omega$ in spherical polar coordinates and assume no azimuthal dependence in I_ν,

$$\Rightarrow F_\nu = 2\pi \int_0^\pi I_\nu \cos\theta \sin\theta \, d\theta$$

$$= 2\pi \int_0^{\pi/2} I_\nu^{out} \cos\theta \sin\theta \, d\theta + 2\pi \int_{\pi/2}^{\pi} I_\nu^{in} \cos\theta \sin\theta \, d\theta$$

We substitute I_ν^{out}, I_ν^{in} into the above equation.

$$F_\nu = 2\pi \int_0^{\pi/2} \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sin\theta \, dt_\nu \, d\theta - 2\pi \int_{\pi/2}^{\pi} \int_0^{\tau_\nu} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sin\theta \, dt_\nu d\theta$$

$$\Rightarrow F_\nu = 2\pi \int_{\tau_\nu}^{\infty} S_\nu \int_0^{\pi/2} e^{-(t_\nu - \tau_\nu)\sec\theta} \sin\theta \, d\theta \, dt_\nu - 2\pi \int_0^{\tau_\nu} S_\nu \int_{\pi/2}^{\pi} e^{-(t_\nu - \tau_\nu)\sec\theta} \sin\theta \, d\theta \, dt_\nu$$

where we have assumed S_ν to be isotropic. Let $w = \sec\theta$ and $x = t_\nu - \tau_\nu$

$$\Rightarrow \int_0^{\pi/2} e^{-(t_\nu - \tau_\nu)\sec\theta} \sin\theta \, d\theta = \int_1^{\infty} \frac{e^{-xw}}{w^2} dw$$

Exponential integrals,

$$E_n(x) \equiv \int_1^{\infty} \frac{e^{-xw}}{w^n} dw$$

which is a monotonically diminishing function of x.

$$F_\nu(\tau_\nu) = 2\pi \int_{\tau_\nu}^{\infty} S_\nu E_2(t_\nu - \tau_\nu) dt_\nu - 2\pi \int_0^{\tau_\nu} S_\nu E_2(\tau_\nu - t_\nu) \, dt_\nu$$

for $E_2(\tau_\nu - t_\nu)$ in 2nd integral, let w = - secθ and $x = \tau_\nu - t_\nu$ in inner integral of 2nd term.

The above equation is the basic relation we have sought. Theoretical stellar spectrum to be compared to observation is $F_\nu(0)$

$$F_\nu(0) = 2\pi \int_0^{\infty} S_\nu(t_\nu) E_2(t_\nu) dt_\nu$$

Surface flux is composed of sum of source function at each depth multiplied by extinction factor $E_2(t_\nu)$, appropriate to that depth below the surface and the sum is taken over all depth contributing a significant amount of radiation at the surface. F_ν is the flux per unit area. Total radiation at frequency v is $4\pi R^2 F_\nu$ where R is star's radius.

THE MEAN INTENSITY AND K INTEGRALS

We derive expressions for $\overline{I_\nu}$ and κ_ν as function of τ_ν.

$$\overline{I_\nu}(\tau_\nu) = \frac{1}{2} \int_{\tau_\nu}^{\infty} S_\nu E_1(t_\nu - \tau_\nu) dt_\nu + \frac{1}{2} \int_0^{\tau_\nu} S_\nu E_1(\tau_\nu - t_\nu) \, dt_\nu$$

$$\kappa_\nu(\tau_\nu) = \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu E_3(t_\nu - \tau_\nu)dt_\nu + \frac{1}{2}\int_{0}^{\tau_\nu} S_\nu E_3(\tau_\nu - t_\nu)\, dt_\nu$$

EXPONENTIAL INTEGRALS' PROPERTIES

$$E_n(x) \equiv \int_1^\infty \frac{e^{-xt}}{t^n} dt$$

$$E_n(0) = \int_1^\infty \frac{dt}{t^n} = \frac{1}{n-1}$$

$$\frac{dE_n}{dx} = \int_1^\infty \frac{1}{t^n}\frac{d}{dx}\left(e^{-tx}\right)dt = -\int_1^\infty \frac{e^{-tx}}{t^{n-1}}\, dt$$

$$or \quad \frac{dE_n}{dx} = -E_{n-1}$$

$$Also, \quad nE_{n+1}(x) = e^{-x} - xE_n(x)$$

RADIATIVE EQUILIBRIUM

Conservation of energy must apply to flow of radiation upward through the stellar photosphere. Assume no sources or sinks of energy in photosphere and the energy generated in core of star is simply flowing outward to outer boundary. => Divergence of flux is zero everywhere in the photosphere. Virtually all the calculations we consider are for plane parallel geometry.

$$\Rightarrow \frac{d}{dx}F(x) = 0 \quad \Rightarrow \quad F(x) = F_0 .$$ which is a constant.

F is total energy flux in ergs/per cm^2. In that case where all the energy is carried via radiation $F = \int_0^\infty F_\nu d\nu$

This special case is called radiative equilibrium and first condition for radiative equilibrium is, $\int_0^\infty F_\nu d\nu = F_0$

This must be satisfied at every depth in photosphere. If convection plays a significant role as a mode of energy transport, then

$$\Phi(x) + \int_0^\infty F_\nu d\nu = F_0$$

$\Phi(x)$ is convective flux. Substitute F_ν,

$$\int_0^\infty \left(\int_{\tau_\nu}^\infty S_\nu E_2(t_\nu - \tau_\nu) dt_\nu - \int_0^{\tau_\nu} S_\nu E_2(\tau_\nu - t_\nu)\, dt_\nu \right) d\nu = \frac{F_0}{2\pi}$$

This is known as Milne's 2nd equation. It says that in case if radiative equilibrium, solution of transfer equation is found when an S_ν is known that satisfies this equation.

Other radiative equilibrium condition that have been used are derived from transfer equation,

$$\cos\theta \frac{dI_\nu}{dx} = \kappa_\nu \rho I_\nu - \kappa_\nu \rho S_\nu$$

Integration over solid angle gives,

$$\frac{d}{dx}\oint I_\nu \cos\theta \, d\omega = \kappa_\nu \rho \oint I_\nu d\omega - \kappa_\nu \rho \oint S_\nu d\omega$$

Assuming $\kappa_\nu\rho$ is independent of direction. If S_ν is also independent of direction and we substitute for 1st and 2nd integrals the definition of flux and mean intensity,

$$\Rightarrow \frac{dF_\nu}{dx} = 4\pi\kappa_\nu \rho \overline{I_\nu} - 4\pi\kappa_\nu \rho S_\nu$$

Second integration over v gives

$$\frac{d}{dx}\int_0^\infty F_\nu d\nu = 4\pi\rho\int_0^\infty \kappa_\nu \overline{I_\nu} d\nu - 4\pi\rho\int_0^\infty \kappa_\nu S_\nu d\nu$$

In radiative equilibrium, left hand side LHS = 0,

$$\int_0^\infty \kappa_\nu \overline{I_\nu} d\nu = \int_0^\infty \kappa_\nu S_\nu d\nu$$

Final radiative equilibrium condition obtained in similar way. Multiply transfer equation by cosθ before integrating over solid angle and frequency with result,

$$\int_0^\infty \frac{d\kappa_\nu}{d\tau_\nu} d\nu = \frac{F_0}{4\pi}$$

Milne's equation corresponding to above 2 equations follows from substituting κ_ν and $\overline{I_\nu}$ into them.

The three Milne equation aren't independent. The S_ν that is a solution of one will be solution of all three.

Flux constant $F_0 = \sigma T^4_{eff}$ where T_{eff} is the effective temperature.

THE GREY CASE

Simplification → κ_ν is frequency independent hence the name grey.
Integrate transfer equation over ν,

$$\cos\theta \frac{d}{dx}\int_0^\infty I_\nu d\nu = \rho\int_0^\infty \kappa_\nu I_\nu d\nu - \rho\int_0^\infty \kappa_\nu S_\nu d\nu$$

Define $\kappa_\nu=\kappa$, $I = \int_0^\infty I_\nu d\nu$ and $S = \int_0^\infty S_\nu d\nu$

$$\Rightarrow \cos\theta \frac{dI}{dx} = \kappa\rho I - \kappa\rho S$$

$$\Rightarrow \cos\theta \frac{dI}{d\tau} = I - S$$

Total radiation is described by a single transfer equation.

$$\Rightarrow F = F_0 \; ; \; \bar{I} = S \; ; \; \frac{dK}{d\tau} = \frac{F_0}{4\pi}$$

Milne's equations simplify to,

$$\frac{1}{2}\int_\tau^\infty SE_1(t-\tau)dt + \frac{1}{2}\int_0^\tau SE_1(\tau - t)dt - S = 0$$

$$\int_{\tau}^{\infty} SE_2(t-\tau)dt - \int_{0}^{\tau} SE_2(\tau - t)\,dt = \frac{F_0}{2\pi}$$

$$\frac{d}{d\tau}\int_{\tau}^{\infty} SE_3(t-\tau)dt + \frac{d}{d\tau}\int_{0}^{\tau} SE_3(\tau - t)\,dt = \frac{F_0}{2\pi}$$

Solution of grey case.

From $\frac{dK}{d\tau} = \frac{F_0}{4\pi} \Rightarrow \kappa(\tau) = \frac{F_0\tau}{4\pi} + const.$

Solution is found by casting both K and F in terms of intensity, which Eddington assumed, as a first approximation, could be represented by a constant inward term and constant outward term.

$I(\tau) = I_{in}(\tau)$ for all $\theta > \pi/2$

$I(\tau) = I_{out}(\tau)$ for all $\theta \le \pi/2$

i.e., at any τ, I is constant over each hemisphere.

$$\Rightarrow \bar{I}(\tau) = \oint \frac{Id\omega}{4\pi} = \frac{1}{2}\int_{0}^{\pi/2} I_{out} \sin\theta\, d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} I_{in} \sin\theta\, d\theta$$

$$= \frac{1}{2} I_{out} \int_{0}^{\pi/2} \sin\theta\, d\theta + \frac{1}{2} I_{in} \int_{\pi/2}^{\pi} \sin\theta\, d\theta$$

or $$\bar{I}(\tau) = \frac{1}{2}\left[I_{out}(\tau) + I_{in}(\tau)\right]$$

Also,

$$F = \oint I \cos\theta \, d\omega = 2\pi \int_0^{\pi/2} I_{out} \cos\theta \sin\theta \, d\theta + 2\pi \int_{\pi/2}^{\pi} I_{in} \cos\theta \sin\theta \, d\theta$$

or $$F(\tau) = \pi\left[I_{out}(\tau) - I_{in}(\tau)\right]$$

Finally,

$$K = \oint I \cos\theta \frac{d\omega}{4\pi} = \frac{1}{2} \int_0^{\pi/2} I_{out} \cos^2\theta \sin\theta \, d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} I_{in} \cos^2\theta \sin\theta \, d\theta$$

or $$K(\tau) = \frac{1}{6}\left[I_{out}(\tau) + I_{in}(\tau)\right]$$

$$\Rightarrow K(\tau) = \frac{1}{3}\bar{I}(\tau)$$

Also, $2\pi\bar{I}(0) = F(0) = F_0$ (evaluated at the outer boundary)

$$\Rightarrow \bar{I}(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right)F_0$$

$$\Rightarrow S(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right)F(0)$$

Source function for grey case as solved using Eddington approximation. It varies linearly with optical depth.

Now we write the frequency integrated source function, S = $\bar{I}$ for the case of pure absorption. Use ν integrated form of Planck's law and along with $F_\nu = \pi I_\nu$

$$\Rightarrow \bar{I}(\tau) = \frac{\sigma}{\pi} T^4$$

$$F_0 = \sigma T_{eff}^4$$

$$\Rightarrow T^4(\tau) = \frac{3}{4}\left(\tau + \frac{2}{3}\right) T^4$$

or $$T(\tau) = \left[\frac{3}{4}\left(\tau + \frac{2}{3}\right)\right]^{1/4} T_{eff}$$

at τ=2/3 T(τ) = T_{eff}

Complete and rigorous solution of grey case leads to

$$\bar{I}(\tau) = \frac{3}{4\pi}\left[\tau + q(\tau)\right] F_0$$

or $$T(\tau) = \left[\frac{3}{4}\left(\tau + q(\tau)\right)\right]^{1/4} T_{eff}$$

q(τ) → slowly varying function → from 0.577 at τ=0 to 0.710 at τ=∞.

CONVECTIVE TRANSPORT

Flux carried by convection can be expressed in terms of average cell ("energy bucket") characteristics : the temperature excess (ΔT), of the cell over its terminal surroundings, the specific heat at constant pressure, C_p ; the cell density, ρ ; and upward velocity, v, of the cell,

$$\Phi = C_p \rho \Delta T v$$

Convection is small contributor to energy flow.
Physical motions of convective cells are important for spectral lines where even small Doppler shifts alter the line profiles. These aerodynamic behaviors are often called turbulence. Momentum of convective cells isn't dissipated when the optical depth of the cells becomes small. Turbulence may extend well into the upper photosphere even though the original cells are in complete radiative exchange with the adjacent gases.

CONDITION FOR CONVECTIVE FLOW

Convective cell must be buoyed upward at each depth if it is to continue to rise. Since density of photosphere is less in higher layers and cell can be expected to stay in pressure equilibrium with its surroundings, it expands as it rises. We calculate density change to see if expanded cell will continue to rise or it will sink.

Assume convective cell behaves adiabatically (no leakage from energy bucket).

γ = ratio of specific heats and P = total pressure

$$\Rightarrow P\rho^{-\gamma} = const.$$

$$\frac{d\log\rho}{d\log P} = \frac{1}{\gamma}$$

Compare this to $\dfrac{d\log\rho}{d\log P}$ in surrounding gas.

For convection, density of cell must decrease at least as rapidly as the photospheric density => For convection,

$$\frac{1}{\gamma} = \frac{d\log\rho}{d\log P}_{cell} > \frac{d\log\rho}{d\log P}_{photosphere}$$

must hold. Use T instead of ρ. Using P = (ρ/μ)kT where μ is mean molecular weight in grams.

$$\log P = \log - \log\mu + \log k + \log T$$

and $$\frac{d\log\rho}{d\log P}_{photosphere} = 1 + \frac{d\log\mu}{d\log P} - \frac{d\log T}{d\log P}$$

$$\Rightarrow \frac{d\log T}{d\log P}_{photosphere} > 1 - \frac{1}{\gamma} + \frac{d\log\mu}{d\log P}$$

This is the condition for convection.

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