Quantum Field Theory in Curved Spacetime and Anomalies

Master's Thesis

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By

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Declaration of Authorship

I, Shivam Garg , declare that this Master's thesis titled, 'Quantum Field Theory in Curved Spacetime and Anomalies' contains results of my own investigations carried out under the guidance of Dr. Sunandan Gangopadhyay. and to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which has formed the basis of award of any other degree or diploma at IISER Kolkata or any other educational institution, except where due acknowledgement is made in the thesis.

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Certificate

This is to certify that the thesis entitled, "Quantum Field Theory in Curved Spacetime and Anomalies" is a bona fide record of work done by Shivam Garg (Roll No. 12MS102) 5 year Integrated BS-MS student, Department of Physical Sciences, under my supervision during August 2016 - April 2017 in partial fulfillment of the requirements for the award of Integrated BS-MS in the Department of Physical Sciences at Indian Institute of Science Education and Research, Kolkata..

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Abstract

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Hawking radiation and Unruh effect are two well known examples of the effect of quantum fields in curved spacetime. Anomalies play an important role in quantum field theory. It has been shown that anomalies can be used to find Hawking temperature. In this Master's thesis, we review Quantum Field Theory in Curved Spacetime (including Unruh and Hawking effects) and then study the application of gravitational anomalies for deriving Hawking radiation in Schwarzschild spacetime.

Contents

Chapter 1

Introduction

General Theory of Relativity is the most successful theory we have at present for describing the gravitational force. Quantum Field Theory is the framework in which quantum mechanics and special relativity are reconciled. Both are extremely successful theories. But it so happens that when we try to quantize gravity, we run into infinities. General relativity in non-renormalizable. To tackle this problem, many approaches such as Loop Quantum Gravity and Supersymmetry have been proposed. Quantum Field Theory in Curved Spacetime is the first order approach to this problem.

This thesis aims at reviewing Quantum Field Theory in surved spacetime and applying the method of anomalies to the problem of calculating Hawking flux. The subject of quantum field theory in curved spacetime is the study of quantum fields propagating in a classical gravitational background. It finds its applications in cases where the spacetime curvature is low enough for quantum gravity effects to be negligible, but too large for Minkowskian quantum field theory. It is not a theory of quantum gravity since the background is not quantised. It is a semiclassical theory but it can give us useful pointers as to what the full quantum theory of gravity must be. Even though it is a first order approximation to the full theory of quantum gravity, we get new and interesting results such as Unruh effect and Hawking Radiation in this regime.

Unruh effect is the following surprising effect: an accelerated observer moving through a Minkowski vacuum will register a thermal spectrum of particle excitations. In layman terms, if one waves a thermometer in empty space (vacuum) then one will measure a certain temperature. This temperature is the Unruh temperature. Hawking Radiation is the prediction that black holes radiate. In classical theory, the gravitational strength of a black hole is so strong that not even a photon can escape it. But as we'll see, black holes do indeed radiate photons which also reduces their mass (black holes evaporate).

Symmetries play an important role in quantum field theory. A symmetry of the classical action is a transformation of the fields that leaves the action invariant, e.g., Lorentz, Poincare transformation. An anomaly is a conflict between a symmetry of the classical action and corresponding symmetry in quantum theory. For example, for a conformally invariant Lagrangian, the trace of the EMT tensor is 0. But in the quantum theory the trace acquires a non-zero value (during renormalization). This is known as the trace/conformal anomaly. This thesis studies the calculation of the Hawking flux from a black hole using the anomaly equation.

The thesis is divided into 4 chapters. Chapter 2 briefly reviews the important concepts from general relativity and quantum field theory is flat spacetime. Chapter 3 introduces the method of graduating from flat to curved spacetime. Unruh effect and the Hawking effect are covered in this chapter. Chapter 4 introduces anomalies and investigates the calculation of Hawking flux from gravitational anomalies. Chapter 5 gives a short conclusion of the thesis. Unless otherwise cited, most of the discussion in this thesis follows [1].

Chapter 2

Review - General Relativity and Quantum Field Theory

In this chapter, we briefly review the main concepts of general relativity and quantum field theory in flat spacetime.

2.1 General Relativity

General Relativity is currently the best description we have for explaining the force of gravity. Einstein's equations of classical general relativity (in the absence of matter) can be derived from the Einstein-Hilbert action:

$$S^{grav} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+2\Lambda) \tag{2.1}$$

where R is the Ricci curvature scalar and Λ is the cosmological constant.

On varying the action with respect to $g^{\alpha\beta}$ and setting it to zero, we get the vacuum Einstein's equations:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0 \tag{2.2}$$

If we have matter fields ϕ_i present, the total action becomes $S = S^{grav} + S^m$. On setting it's variation to zero and defining $T_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta S^m}{\delta g^{\alpha\beta}}$ we write:

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \tag{2.3}$$

The tensor T_{ab} is symmetric and covariantly conserved w.r.t general coordinate transformation. $T^{a}_{b;a} = 0$

2.2 Quantum Field Theory in Flat Spacetime

A free field can be treated as a set of infinitely many harmonic oscillators $q_i(t) \Leftrightarrow \phi_{\mathbf{x}}(t)$ attached to each point $\mathbf{x}$. Here $\mathbf{x}$ plays the role of index labelling the oscillator, same as discrete index i. The action for a scalar field is written in analogy with the action for describing N harmonic oscillators.

$$S[\phi] = \frac{1}{2}\int dt \left[\int d^3\mathbf{x}\dot{\phi}^2(\mathbf{x},t) - \int d^3\mathbf{x}d^3\mathbf{y}\phi(\mathbf{x},t)\phi(\mathbf{y},t)M(\mathbf{x},\mathbf{y})\right] \tag{2.4}$$

The action must be invariant w.r.t. the Lorentz transformations (boosts and rotations) and to the spacetime translations (together, the Poincare group). The simplest Poincare invariant action for a real scalar field if obtained for

$$M(\mathbf{x},\mathbf{y}) = \left[-\Delta_{\mathbf{x}} + m^2\right]\delta(\mathbf{x}-\mathbf{y}) \tag{2.5}$$

The action becomes

$$S[\phi] = \frac{1}{2}\int d^3\mathbf{x}dt\left[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right] = \frac{1}{2}\int d^4x\left[\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2\right] \tag{2.6}$$

where $\eta^{\mu\nu} = diag(1,-1,-1,-1)$. The action is manifestly translationaly invariant and can be shown to be Lorentz invariant. If we find the equation of motion for the above action, we get

$$\ddot{\phi}(\mathbf{x},t) - \Delta\phi(\mathbf{x},t) + m^2\phi(\mathbf{x},t) = 0 \tag{2.7}$$

The oscillators $\phi(\mathbf{x},t)$ are coupled (due to the presence of second derivative in space). To decouple them, we use the Fourier transform

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}(t) \tag{2.8}$$

This converts the equation of motion into

$$\ddot{\phi}_{\mathbf{k}}(t) + \omega_k^2\phi_{\mathbf{k}}(t) = 0 \quad \text{where} \quad \omega_k = \sqrt{k^2+m^2} \tag{2.9}$$

For quantizing this scalar field, we need to cast the theory into the Hamiltonian formalism. The canonical momenta are defined as the functional derivatives of the Lagrangian w.r.t. the generalized velocities $\dot{\phi} = \frac{\partial \phi}{\partial t}$,

$$\pi(\mathbf{x}, t) = \frac{\delta L[\phi]}{\partial \dot{\phi}(\mathbf{x}, t)} = \dot{\phi}(\mathbf{x}, t) \tag{2.10}$$

The Hamiltonian is then

$$H = \int d^3\mathbf{x}\ \pi\dot{\phi} - L = \frac{1}{2}\int d^3\mathbf{x}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \tag{2.11}$$

and the Hamilton's equations of motions are

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = \frac{\delta H}{\delta \pi(\mathbf{x}, t)} = \pi(\mathbf{x}, t) \qquad \frac{\partial \pi(\mathbf{x}, t)}{\partial t} = -\frac{\delta H}{\delta \phi(\mathbf{x}, t)} = \Delta\phi(\mathbf{x}, t) - m^2\phi(\mathbf{x}, t) \tag{2.12}$$

For quantising the scalar field, we introduce the operators $\hat{\phi}(\mathbf{x}, t)$ and $\hat{\pi}(\mathbf{x}, t)$ and impose the standard equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}, t)\right] = i\delta(\mathbf{x} - \mathbf{y}); \qquad \left[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{y}, t)\right] = \left[\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)\right] = 0 \tag{2.13}$$

We can introduce the creation and annihilation operators,

$$\hat{a}^-_{\mathbf{k}}(t) \equiv \sqrt{\frac{\omega_k}{2}}\left(\hat{\phi}_{\mathbf{k}} + \frac{i\hat{\pi}_{\mathbf{k}}}{\omega_k}\right); \quad \hat{a}^+_{\mathbf{k}}(t) \equiv \left(\hat{a}^-_{\mathbf{k}}(t)\right)^\dagger = \sqrt{\frac{\omega_k}{2}}\left(\hat{\phi}_{-\mathbf{k}} - \frac{i\hat{\pi}_{-\mathbf{k}}}{\omega_k}\right) \tag{2.14}$$

which satisfy the commutation relations

$$\left[\hat{a}^-_{\mathbf{k}}(t), \hat{a}^+_{\mathbf{k}'}(t)\right] = \delta(\mathbf{k} - \mathbf{k}') \quad ; \left[\hat{a}^-_{\mathbf{k}}(t), \hat{a}^-_{\mathbf{k}'}(t)\right] = \left[\hat{a}^+_{\mathbf{k}}(t), \hat{a}^+_{\mathbf{k}'}(t)\right] = 0 \tag{2.15}$$

We can also solve for $\hat{a}^{\pm}$ as a function of time using Hamilton's equation of motion. We use the time independent part of the solution (which also satisfy the same commutation relations) in the subsequent analysis. We can build up the Hilbert space by postulating the existence of the vacuum state $|0\rangle$ which is annihilated by all operators $\hat{a}^-_{\mathbf{k}}$, i.e., $\hat{a}^-_{\mathbf{k}}|0\rangle = 0\ \forall\ \mathbf{k}$. The quantum states are constructed by applying the creation operators with the required momentum repeatedly on the vacuum state. The basis of the Hilbert space can be formed from considering all possible choices of required momentum and required occupation number.

Alternatively, we can begin directly with the mode expansion of the quantum field $\hat{\phi}(\mathbf{x}, t)$

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[v^*_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}^-_{\mathbf{k}} + v_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^+_{\mathbf{k}}\right] \tag{2.16}$$

and postulate the commutation relations for the time-independent operators $\hat{a}_{\mathbf{k}}^-$ and $\hat{a}_{\mathbf{k}}^+$,

$$\left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^+\right] = \delta(\mathbf{k} - \mathbf{k}') \quad ; \left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^-\right] = \left[\hat{a}_{\mathbf{k}}^+, \hat{a}_{\mathbf{k}'}^+\right] = 0 \tag{2.17}$$

From the equations of motion, we find that the mode functions $v_{\mathbf{k}}(t)$ satisfy the equation

$$\ddot{v}_{\mathbf{k}} + \omega_k^2 v_{\mathbf{k}} = 0 \tag{2.18}$$

where $\omega_k^2 = k^2 + m^2$. Substituting the mode expansion for $\hat{\phi}$ and $\hat{\pi}$ into the canonical commutation relations, we find that the canonical commutation relations are compatible with (2.17) only if the normalization conditions

$$\dot{v_{\mathbf{k}}}(t) v_{\mathbf{k}}^*(t) - v_{\mathbf{k}}(t) \dot{v}_{\mathbf{k}}^*(t) = 2i \tag{2.19}$$

are satisfied. Substituting the general solution of equation (2.18)

$$v_{\mathbf{k}}(t) = \frac{1}{\sqrt{\omega_k}} \left(\alpha_{\mathbf{k}} e^{i\omega_k t} + \beta_{\mathbf{k}} e^{-i\omega_k t}\right) \tag{2.20}$$

into the normalisation conditions, we find that

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{2.21}$$

This condition is not enough to determine $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$. Therefore the operators $\hat{a}_{\mathbf{k}}^-$ and $\hat{a}_{\mathbf{k}}^+$ are not yet unambiguously defined. This is the fact we use further to demonstrate that we have two different sets of mode functions for the same field in curved spacetime. In this case (flat spacetime), we can use the Hamiltonian to find another condition which will help fix $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$. Another thing to note is that the modes $v_k^*(t) \propto e^{-i\omega_k t}$ and $v_k(t) \propto e^{i\omega_k t}$ are known as the positive and negative frequency modes respectively.

Chapter 3

Quantum Field Theory in Curved Spacetime

In this chapter, we describe the basics of Quantum Field Theory in curved spacetime. QFT in CST describes quantum fields in the presence of gravitational fields in regimes where the quantum nature of gravity does not play an important role. The back-reaction of the quantum fields on the metric is usually neglected. The spacetime is described, as in general relativity, by a manifold, M, on which a Lorentz metric, g_{ab}, is defined. We will find interesting results such as Unruh effect (a uniformly accelerating observer moving through a vacuum state measures a finite temperature) and Hawking radiation (black holes radiate).

In flat spacetime, Lorentz invariance plays an important role. It allows us to identify a unique vacuum state. However, in curved spacetime, we do not have Lorentz symmetry. In general, there does not exist a unique vacuum state in a curved spacetime. As a result, the concept of particles becomes ambiguous.

3.1 Quantum Driven Harmonic Oscillator

In Quantum Field Theory in Curved Spacetime, a time-varying background metric leads to particle production. The driven harmonic oscillator is a lower dimensional classical analog for this.

3.1.1 Lagrangian and Quantisation

The Lagrangian for the driven harmonic oscillator is

$$L(t, q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t)q \tag{3.1}$$

The corresponding Hamiltonian and Hamilton's equation of motions are

$$H(p, q) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} - J(t)q; \qquad \dot{q} = p, \quad \dot{p} = -\omega^2 q + J(t) \tag{3.2}$$

We promote the classical variables to operators and impose the commutation relation $[\hat{q}, \hat{p}] = i$. Introducing the creation and annihilation operators (as done in previous chapter), we can find the equations of motion for the operators $\hat{a}^{\pm}(t)$. On solving for $\hat{a}^{\pm}(t)$, we find,

$$\hat{a}^{\pm}(t) = \left[\hat{a}^{\pm}_{in} \mp \frac{i}{2\omega}\int_0^t e^{\mp i\omega t'} J(t')dt'\right] e^{\pm i\omega t} \tag{3.3}$$

where $\hat{a}^{\pm}_{in}$ are operator-valued constants of integration.

3.1.2 Particle production

Assume that J(t) is non-vanishing for $t \in [0, T]$ only. We can demarcate two regions now, the "in" region $t < 0$ and the "out" region $t > T$. In both regions, the oscillator is unperturbed and the form of J(t) is irrelevant as long as it is well-behaved. We have to find the relation between the "in" states and the "out" states and show that particle production indeed takes place. The "in" and "out" regions can be considered to be analogous to flat spacetime and the driving force region can be considered as curved spacetime.

It follows from (3.3) that in the "in" region we have $\hat{a}^{\pm}(t) = \hat{a}^{\pm}_{in}e^{\pm i\omega t}$ and correspondingly in out region we have $\hat{a}^{\pm}(t) = \hat{a}^{\pm}_{out}e^{\pm i\omega t}$ where

$$\hat{a}^{-}_{out} \equiv \hat{a}^{-}_{in} + \frac{i}{\sqrt{2\omega}}\int_0^T e^{i\omega t'} J(t')dt' \equiv \hat{a}^{-}_{in} + J_0, \qquad \hat{a}^{+}_{out} = \hat{a}^{+}_{in} + J_0^* \tag{3.4}$$

We have 2 sets of creation and annihilation operators. We can construct the Hilbert space using them in both the "in" and "out" regions. The two annihilation operators, $\hat{a}^{-}_{in}$ and $\hat{a}^{-}_{out}$ define two different vacuum states - the "in" vacuum state $|0_{in}\rangle$ and the "out" vacuum state $|0_{out}\rangle$. These states are the lowest-energy states for $t < 0$ and for $t > T$ respectively. But the physical interpretation of both the states is different. The state $|0_{in}\rangle$ is an eigenstate of $\hat{a}^{-}_{out}$

$$\hat{a}^{-}_{out}|0_{in}\rangle = \left(\hat{a}^{-}_{in} + J_0\right)|0_{in}\rangle = J_0|0_{in}\rangle \tag{3.5}$$

The eigenstates of the annihilation operator with nonzero eigenvalues are called coherent states. Also $\hat{a}_{in}^{-}\left|0_{out}\right\rangle = -J_0\left|0_{out}\right\rangle$. Note that we are working in the Heisenberg picture where the operators evolve with time and quantum states are time independent. If we start out in the vacuum state $\left|0_{in}\right\rangle$, we stay in that state for all time, but the physical interpretation of the state changes with time. $\left|0_{in}\right\rangle$ is the lowest energy state for $t < 0$, but we will see that due to the external force J(t), the energy of the system changes and hence $\left|0_{in}\right\rangle$ is no longer the lowest energy state for $t > T$.
We can find the vector $\left|0_{in}\right\rangle$ in terms of the "out" states (since the "out" states form a complete basis in the Hilbert space). We find

$$\left|0_{in}\right\rangle = exp\left[-\frac{1}{2}\left|J_0\right|^2 + J_0\hat{a}_{out}^{+}\right]\left|0_{out}\right\rangle \tag{3.6}$$

The occupation number operator $\hat{N}(t) = \hat{a}^{+}(t)\hat{a}^{-}(t)$ has expectation value

$$\left\langle 0_{in}\right|\hat{N}(t)\left|0_{in}\right\rangle = \begin{cases} 0 & for\ t < 0 \\ \left|J_0\right|^2 & for\ t > T \end{cases} \tag{3.7}$$

Hence, the energy expectation value gets shifted

$$\left\langle 0_{in}\right|\hat{H}(t)\left|0_{in}\right\rangle = \begin{cases} \frac{\omega}{2} & for\ t < 0 \\ \left(\frac{1}{2} + \left|J_0\right|^2\right)\omega & for\ t > T \end{cases} \tag{3.8}$$

The energy of the oscillator in the "out" region becomes larger than the zero-point energy. This can be interpreted as production of particles after the application of the force J(t).

3.2 From flat to curved spacetime

The simplest relativistically invariant Lagrangian density for a real scalar field $\phi(x)$ in a flat spacetime is:

$$\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) \tag{3.9}$$

where $\eta^{\mu\nu}$ is the Minkowski metric $V(\phi)$ describes the self-interaction of the field. For generalising the Lagrangian from flat to curved spacetime with an arbitrary metric $g_{\mu\nu}$, we have to:

- replace $\eta_{\mu\nu}$ with the metric $g_{\mu\nu}$
- replace ordinary derivatives by covariant derivatives

- use the covariant volume element $d^4x\sqrt{-g}$ where $g \equiv \det g_{\mu\nu}$ instead of the usual volume element $d^3\mathbf{x}dt$

The resulting action,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right] \tag{3.10}$$

depends explicitly on $g_{\mu\nu}$ and describes a scalar field which is minimally coupled to gravity.

3.2.1 Nonminimal and conformal couplings

The action can contain additional terms which directly couple the fields to gravity via the curvature tensor $R_{\mu\nu\rho\sigma}$. Such couplings are called nonminimal. The simplest action for a nonminimally coupled scalar field is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) - \frac{\xi}{2}R\phi^2\right] \tag{3.11}$$

where R is the Ricci curvature scalar and ξ is a constant parameter. The additional term gives rise to a "mass" correction which is proportional to the scalar curvature. With V=0 and $\xi = 1/6$ this action has another symmetry, i.e., the action is invariant under conformal transformations,

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \tag{3.12}$$

where the conformal factor is an arbitrary function of spacetime.

Conformal invariance is important because as we will see later, in conformally flat spacetimes, where the metric can be written as $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, the field decouples from gravity, since its action is equivalent to an action in flat space.

The equation of motion for this action can be written as

$$\phi_{;\alpha}^{;\alpha} + \frac{\partial V}{\partial\phi} + \xi R\phi = 0 \tag{3.13}$$

3.3 Fields in FLRW models

We only consider the class of a scalar field minimally coupled to a spatially flat FLRW metric. The metric, in coordinates that make its symmetries manifest, is

$$ds^2 = dt^2 - a^2(t)\delta_{ab}dx^a dx^b \tag{3.14}$$

Define a new coordinate, the conformal time

$$\eta(t) \equiv \int^t \frac{dt}{a(t)} \tag{3.15}$$

in terms of which the conformal equivalence of the metric to the Minkowski metric $\eta_{\mu\nu}$ becomes manifest:

$$ds^2 = a^2(\eta)\left[d\eta^2 - \delta_{ab}dx^a dx^b\right] = a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu \tag{3.16}$$

The action for a real minimally coupled massive scalar field $\phi(x)$ in a curved spacetime is

$$S = \frac{1}{2}\int \sqrt{-g}d^4x\left[g^{ab}\phi_{,a}\phi_{,b} - m^2\phi^2\right] \tag{3.17}$$

Substituting $g^{ab} = a^{-2}\eta^{ab}$ and $\sqrt{-g} = a^4$ we get

$$S = \frac{1}{2}\int d^3\mathbf{x}d\eta a^2\left[\phi'^2 - (\nabla\phi)^2 - m^2a^2\phi^2\right] \tag{3.18}$$

where the prime ' denotes derivative w.r.t. conformal time. Defining the auxiliary field $\chi \equiv a(\eta)\phi$ we rewrite the action in terms of χ (eliminating the total derivative terms)

$$S = \frac{1}{2}\int d^3\mathbf{x}d\eta\left[\chi'^2 - (\nabla\chi)^2 - \left(m^2a^2 - \frac{a''}{a}\right)\chi^2\right] \tag{3.19}$$

The variation of the above action w.r.t. χ gives the equation of motion

$$\chi'' - \Delta\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0 \tag{3.20}$$

This equation is formally equivalent to that of a Klein-Gordon field in Minkowski spacetime, except that the effective mass becomes time-dependent

$$m^2_{eff}(\eta) = m^2a^2 - \frac{a''}{a} \tag{3.21}$$

Thus the problem is mathematically equivalent to the problem of quantizing a free scalar field in Minkowski spacetime. Note that the action is time-dependent, hence the energy of the scalar field is not conserved which leads to particle creation.

We expand the field χ in Fourier modes and substitute the exapnsion into the equation of motion to find that the Fourier modes $\chi_{\mathbf{k}}(\eta)$ satisfy a set of decoupled ordinary differential equations

$$\chi''_{\mathbf{k}} + \omega^2_k(\eta)\chi_{\mathbf{k}} = 0 \tag{3.22}$$

where

$$\omega_k^2(\eta) = k^2 + m_{eff}^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''}{a} \tag{3.23}$$

Since $\omega_k^2(\eta)$ depends only on $k \equiv |\mathbf{k}|$, the general solution for $\chi_{\mathbf{k}}$ may be written as

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}} \left[a_{\mathbf{k}}^- v_k^*(\eta) + a_{-\mathbf{k}}^+ v_k(\eta) \right] \tag{3.24}$$

where $v_k(\eta)$ and $v_k^*(\eta)$ are two linearly independent solutions of second order differential equation. The two complex constants of integration $a_{\mathbf{k}}^{\pm}$ can depend on the direction of $\mathbf{k}$ as well. Since the field χ is real, we have $\chi_{\mathbf{k}}^*(\eta) = \chi_{-\mathbf{k}}(\eta)$ which implies that $a_{\mathbf{k}}^+ = (a_{\mathbf{k}}^-)^*$. It can be easily shown that v_k and v_k^* are linearly independent iff their Wronskian

$$W\left[v_k, v_k^*\right] = v_k' v_k^* - v_k v_k^{*\prime} = 2i\mathrm{Im}(v' v^*) \tag{3.25}$$

is nonzero. Also, eq. (3.22) implies that the Wronskian will be time-independent. Hence, if W is nonzero, we can always normalize v_k such that $\mathrm{Im}(v'v^*) = 1$. In this case the complex solution $v_k(\eta)$ is called a mode function. Considering all this, we get

$$\chi(\mathbf{x}, \eta) = \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[a_{\mathbf{k}}^- v_k^*(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^+ v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \tag{3.26}$$

The field χ is quantized by imposing the standard equal-time commutation relations on the field operator $\hat{\chi}$ and its canonically conjugate momentum $\hat{\pi} \equiv \hat{\chi}'$

$$[\hat{\chi}(\mathbf{x}, \eta), \hat{\pi}(\mathbf{y}, \eta)] = i\delta(\mathbf{x} - \mathbf{y}) \tag{3.27}$$

$$[\hat{\chi}(\mathbf{x}, \eta), \hat{\chi}(\mathbf{y}, \eta)] = [\hat{\pi}(\mathbf{x}, \eta), \hat{\pi}(\mathbf{y}, \eta)] = 0 \tag{3.28}$$

The Hamiltonian is given by

$$\hat{H}(\eta) = \frac{1}{2} \int d^3\mathbf{x} \left[\hat{\pi}^2 + (\nabla\hat{\chi})^2 + m_{eff}^2(\eta)\hat{\chi}^2 \right] \tag{3.29}$$

Alternatively, one can impose the commutation relations on the constants of integration $a_{\mathbf{k}}^{\pm}$

$$\left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^+\right] = \delta(\mathbf{k} - \mathbf{k}') \qquad \left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^-\right] = \left[\hat{a}_{\mathbf{k}}^+, \hat{a}_{\mathbf{k}'}^+\right] = 0 \tag{3.30}$$

together with the constraints that the mode function satisfy eq. (3.22) and the normalization condition $\mathrm{Im}(v_k' v_k^*) = 1$. $a_{\mathbf{k}}^{\pm}$ are now interpreted as the creation and annihilation operators.

Here in lies the catch. Until we select the particular mode functions $v_k(\eta)$, the states constructed using the creation and annihilation operators have an ambiguous physical interpretation. The normalization condition is not enough to completely specify the complex solutions $v_k(\eta)$ of the second order differential equation (3.22). The functions

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta) \tag{3.31}$$

with the condition $|\alpha_k|^2 - |\beta_k|^2 = 1$ can be used as mode functions instead of $v_k(\eta)$.

The field operator can be expanded in terms of the mode functions $u_k(\eta)$ as

$$\hat{\chi}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} u_k^*(\eta) \hat{b}_{\mathbf{k}}^- + e^{-i\mathbf{k}\cdot\mathbf{x}} u_k(\eta) \hat{b}_{\mathbf{k}}^+ \right] \tag{3.32}$$

where $b_{\mathbf{k}}^{\pm}$ are another set of creation and annihilation operators satisfying the commutation relations. We can compare the two mode expansions to find the *Bogoliubov transformation*

$$\hat{a}_{\mathbf{k}}^- = \alpha_k^* \hat{b}_{\mathbf{k}}^- + \beta_k \hat{b}_{-\mathbf{k}}^+, \qquad \hat{a}_{\mathbf{k}}^+ = \alpha_k \hat{b}_{\mathbf{k}}^+ + \beta_k^* \hat{b}_{-\mathbf{k}}^- \tag{3.33}$$

or inverting the equations to obtain $b_{\mathbf{k}}^{\pm}$

$$\hat{b}_{\mathbf{k}}^- = \alpha_k \hat{a}_{\mathbf{k}}^- - \beta_k \hat{a}_{-\mathbf{k}}^+, \qquad \hat{b}_{\mathbf{k}}^+ = \alpha_k^* \hat{a}_{\mathbf{k}}^+ - \beta_k^* \hat{a}_{-\mathbf{k}}^- \tag{3.34}$$

Each set of creation and annihilation operators define their respective vacuums and their respective Hilbert spaces. It is however not necessary that the vacuum state is same for both set of operators. This can be shown by calculating the number operator for $\hat{a}_{\mathbf{k}}^{\pm}$ in the $\hat{b}_{\mathbf{k}}^{\pm}$ vacuum state

$$\left\langle {}_{(b)}0 \right| \hat{N}_{\mathbf{k}}^{(a)} \left| {}_{(b)}0 \right\rangle = \left\langle {}_{(b)}0 \right| \hat{a}_{\mathbf{k}}^+ \hat{a}_{\mathbf{k}}^- \left| {}_{(b)}0 \right\rangle = |\beta_k|^2 \delta^{(3)}(0) \tag{3.35}$$

The divergent factor $\delta^{(3)}(0)$ is a result of quantization in infinite volume and hence the mean density of the a-particles in the mode $\mathbf{k}$ is

$$n_{\mathbf{k}} = |\beta_k|^2 \tag{3.36}$$

which shows that there exist a-particles in b-vacuum state. Next we are going to demonstrate two related effects.

3.4 Unruh effect

The notion of particles depends on the definition of the positive frequency modes, which an inertial observer defines w.r.t. the time t of some inertial reference frame. An accelerating observer, however, defines the positive-frequency modes w.r.t. the proper time (the time in the frame in which the observer is at rest). Hence, two observers, one inertial and one accelerated, will not agree on the number and nature of particles they detect when they are observing the same region of spacetime. Fulling [2], Davies [3] and Unruh [4] showed that an accelerating observer will observe a thermal bath of particles whereas an inertial observer observes only vacuum. This effect is called Unruh effect and the temperature of the observed thermal bath, the Unruh Temperature, is proportional to the acceleration

$$T \equiv \frac{a}{2\pi} \tag{3.37}$$

We will now show this using a massless scalar field and assume that the observer moves with a constant acceleration in a 1+1-dimensional spacetime. The idea is to determine the trajectory of the accelerated observer in an inertial frame, construct an accelerated comoving frame and then solve the wave equation and compare the notion of particles in both coordinate frames.

3.4.1 Trajectory of an accelerated observer

Consider the two-dimensional Minkowski spacetime

$$ds^2 = dt^2 - dx^2 = \eta_{ab} dx^a dx^b \tag{3.38}$$

The 2-velocity defined using the proper time $u^a = dx^a/d\tau$ satisfies the normalisation condition $u^i u_i = 1$. The condition for constant acceleration can be covariantly stated as $a^i a_i = -a^2$ where a^i is the 2-acceleration and a is the constant acceleration.

The calculation is simplified if we use the lightcone coordinates. The inertial lightcone coordinates are defined as

$$u \equiv t - x \qquad\qquad v \equiv t + x \tag{3.39}$$

so the metric becomes

$$ds^s = du dv \tag{3.40}$$

Notice that the coordinate transformation

$$u \longrightarrow \tilde{u} = \alpha u, v \longrightarrow \tilde{v} = \frac{v}{\alpha} \tag{3.41}$$

with α=constant leaves the metric invariant and hence is a Lorentz transformation.

The trajectory can be described in lightcone coordinates by

$$x^\alpha(\tau) = (u(\tau), v(\tau)) \tag{3.42}$$

Using the normalisation conditions for proper velocity and proper acceleration, we find that

$$\dot{u}(\tau)\dot{v}(\tau) = 1 \qquad \ddot{u}(\tau)\ddot{v}(\tau) = -a^2 \tag{3.43}$$

which after solving yields

$$v(\tau) = \frac{A}{a}e^{a\tau} + B \qquad u(\tau) = -\frac{1}{Aa}e^{-a\tau} + C \tag{3.44}$$

Performing a Lorentz transformation, we can set $A = 1$ and shifting the origin of the corresponding inertial frame we can set $B = C = 0$. Therefore the trajectory becomes

$$u(\tau) = -\frac{1}{a}e^{-a\tau} \qquad v(\tau) = \frac{1}{a}e^{a\tau} \tag{3.45}$$

Going back to the original Minkowski coordinates t and x we have

$$t(\tau) = \frac{1}{a}\sinh a\tau \qquad x(\tau) = \frac{1}{a}\cosh a\tau \tag{3.46}$$

The worldline of an accelerated observer is the right branch of the hyperbola $x^2 - t^2 = a^{-2}$. The observer arrives from infinity, momentarily comes to rest at $x = a^{-1}$ and then accelerates back to infinity.

3.4.2 Accelerated comoving coordinates

Now, we find a frame (ξ^0, ξ^1) comoving with the accelerating observer. The observer is at rest at $\xi^1 = 0$ and ξ^0 coincides with the proper time τ along the observer's worldline. We would also like the metric to be conformally flat to simplify the quantization of fields.

$$ds^2 = \Omega^2(\xi^0, \xi^1)\left[(d\xi^0)^2 - (d\xi^1)^2\right] \tag{3.47}$$

where the scale factor $\Omega(\xi^0, \xi^1)$ is yet to be determined. The lightcone coordinates of the comoving frame are

$$\tilde{u} \equiv \xi^0 - \xi^1 \qquad \tilde{v} \equiv \xi^0 + \xi^1 \tag{3.48}$$

in which the metric becomes

$$ds^2 = \Omega^2(\tilde{u}, \tilde{v})d\tilde{u}d\tilde{v} \tag{3.49}$$

and the observer's worldline

$$\xi^0(\tau) = \tau \qquad \xi^1(\tau) = 0 \tag{3.50}$$

becomes

$$\tilde{v}(\tau) = \tilde{u}(\tau) = \tau \tag{3.51}$$

ξ^0 is the proper time w.r.t. the observer's location, hence

$$\Omega^2(\tilde{u} = \tau, \tilde{v} = \tau) = 1 \tag{3.52}$$

Now, eq.(3.40) and eq. (3.49) describe the same Minkowski spacetime in different coordinate systems and hence

$$ds^2 = dudv = \Omega^2(\tilde{u}, \tilde{v})d\tilde{u}\tilde{v} \tag{3.53}$$

The functions $u(\tilde{u}, \tilde{v})$ and $v(\tilde{u}, \tilde{v})$ can depend on only one of the two arguments, otherwise there will be terms such as $d\tilde{u}^2$ in the latter equality in the previous equation. We choose

$$u = u(\tilde{u}) \qquad\qquad v = v(\tilde{v}) \tag{3.54}$$

We shall now determine the functions $u(\tilde{u})$ and $v(\tilde{v})$. If we consider the observer's trajectory in two coordinate systems and use equations 3.45 and 3.51, we can solve for $u(\tilde{u})$ and $v(\tilde{v})$. We have

$$u = C_1 e^{-a\tilde{u}} \tag{3.55}$$

and

$$v = C_2 e^{a\tilde{v}} \tag{3.56}$$

where C_1 and C_2 are restricted by eq. 3.52. Taking $C_2 = -C_1$ we obtain

$$u = -\frac{1}{a}e^{-a\tilde{u}} \qquad v = \frac{1}{a}e^{a\tilde{v}} \tag{3.57}$$

and the line interval becomes

$$ds^2 = dudv = e^{a(\tilde{v}-\tilde{u})}d\tilde{u}d\tilde{v} \tag{3.58}$$

The metric in accelerated frame becomes

$$ds^2 = e^{2a\xi_1}\left[(d\xi^0)^2 - (d\xi^1)^2\right] \tag{3.59}$$

which is the Rindler metric.

We can show that the coordinates ξ^0 and ξ^1 cover only the right wedge of the 1+1 dimensional Minkowski spacetime. Hence this coordinate system is incomplete. The accelerated observer cannot observe more than a^{-1} in the direction opposite to the acceleration. No comoving frame with an accelerating observer can cover the entire Minkowski spacetime. The lightcone t=x plays the role of an event horizon.

3.4.3 Quantum Fields in inertial and accelerated frames

Now consider a massless scalar field in 1+1 dimensional spacetime with minimal coupling to gravity. The action is given by

$$S[\phi] = \frac{1}{2}\int g^{ab}\phi_{,a}\phi_{,b}\sqrt{-g}d^2x \tag{3.60}$$

It can be easily shown that this action is conformally invariant (since the determinant $\sqrt{-g}$ changes by a factor Ω^2 and the metric changes by a factor Ω^{-2} which cancel out in the action). Hence the action looks same in both the inertial and accelerated frames.

$$S = \frac{1}{2}\int \left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] dtdx \tag{3.61}$$

$$= \frac{1}{2}\int \left[(\partial_{\xi^0}\phi)^2 - (\partial_{\xi^1}\phi)^2\right] dtdx \tag{3.62}$$

In terms of lightcone coordinates,

$$S = 2\int \partial_u\phi\partial_v\phi dudv = 2\int \partial_{\tilde{u}}\phi\partial_{\tilde{v}}\phi d\tilde{u}d\tilde{v} \tag{3.63}$$

The field equations

$$\partial_u\partial_v\phi = 0 \qquad\qquad \partial_{\tilde{u}}\partial_{\tilde{v}}\phi = 0 \tag{3.64}$$

have the solutions,

$$\phi(u,v) = A(u) + B(v), \qquad \phi(\tilde{u},\tilde{v}) = \tilde{A}(\tilde{u}) + \tilde{B}(\tilde{v}) \tag{3.65}$$

where $A, \tilde{A}, B, \tilde{B}$ are arbitrary smooth functions. Particularly,

$$\phi \propto e^{-i\omega u} = e^{-i\omega(t-x)} \tag{3.66}$$

describes a right-moving, positive-frequency mode w.r.t. the Minkowski time t, while

$$\phi \propto e^{-i\omega\tilde{u}} = e^{-i\Omega(\xi^0-\xi^1)} \tag{3.67}$$

describes a right-moving positive frequency mode w.r.t. the proper time $\tau = \xi^0$. The solutions $\phi \propto e^{-i\omega v}$ and $\phi \propto e^{-i\Omega\tilde{v}}$ describe left-moving modes. The left and right moving modes do not affect each other and can be considered separately. In the right wedge of the Minkowski spacetime where both coordinate systems overlap, we can write the mode expansion for the field operator $\hat{\phi}$ as

$$\hat{\phi} = \int \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega u} \hat{a}_\omega^- + e^{i\omega u} \hat{a}_\omega^+] + \text{(left-moving modes)} \tag{3.68}$$

$$= \int \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega \tilde{u}} \hat{b}_\Omega^- + e^{i\Omega \tilde{u}} \hat{b}_\Omega^+] + \text{(left-moving modes)} \tag{3.69}$$

where $\hat{a}_\omega^\pm, \hat{b}_\Omega^\pm$ both sets of operators satisfy the standard commutation relations. We can associate a vacuum state with each set of creation and annihilation operators.

The Minkowski vacuum $|0_M\rangle$ is the zero eigenvector of all annihilation operators $\hat{a}_\omega^-$

$$\hat{a}_\omega^- \left|0_M\right\rangle = 0 \tag{3.70}$$

Similarly, Rindler vacuum is the zero eigenvector of all annihilation operators $\hat{b}_\Omega^-$

$$\hat{b}_\Omega^- \left|0_R\right\rangle = 0 \tag{3.71}$$

The Minkowski vacuum is the physical vacuum which is defined w.r.t. the inertial observer wheras the Rindler vacuum is defined w.r.t. the accelerating observer. An inertial observer detects no particles in the Minkowski vacuum. Similarly, an accelerating observer detects no particles in the Rindler vacuum. But to the accelerating observer the Minkowski vacuum will appear to be a state with particles. This is the Unruh effect. We now calculate the occupation number of the Rindler particles in the Minkowski vacuum state.

3.4.4 Unruh temperature

We can relate the operators $\hat{a}^\pm$ and $\hat{b}^\pm$ using the Bogoliubov transformation

$$\hat{b}_\Omega^- = \int d\omega \left[\alpha_{\Omega\omega} \hat{a}_\omega^- - \beta_{\Omega\omega} \hat{a}_\omega^+\right] \tag{3.72}$$

The normalization condition for the Bogoliubov coefficients is

$$\int d\omega \left(\alpha_{\Omega\omega} \alpha^*_{\Omega'\omega} - \beta_{\Omega\omega} \beta^*_{\Omega'\omega}\right) = \delta(\Omega - \Omega') \tag{3.73}$$

which follows from the compatibility of the commutation relations for the creation and annihilation operators.

Using the Bogoliubov transformation in the mode expansion for $\hat{\phi}$ we obtain a useful relation

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\beta_{\Omega\omega}|^2 \tag{3.74}$$

As earlier, we now compute the occupation number $< \hat{N}_\Omega >$ for the Rindler b-particles in the Minkowski vacuum state.

$$\left\langle \hat{N}_\Omega \right\rangle \equiv \langle 0_M | \, \hat{b}^+_\Omega \hat{b}^-_\Omega \, | 0_M \rangle = \left[exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \delta(0) \tag{3.75}$$

where we have used the Bogoliubov transformation and eq. (3.74). This is interpreted as the mean number of particles with frequency Ω found by the accelerated observer.

The divergent factor $\delta(0)$ is due to the infinite volume of space. If we take the field in a finite box and then quantise it, we can obtain the mean density of particles with frequency Ω as

$$n_\Omega = \left[exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \tag{3.76}$$

We can do a similar calculation for the left-moving modes as well. Thus we see that massless particles detected by the accelerated observer in the Minkowski vacuum obey the Bose-Einstein distribution with the Unruh temperature

$$T \equiv \frac{a}{2\pi} \tag{3.77}$$

3.5 Hawking Radiation

We can do a similar calculation (as done in the previous section) for a black hole background and find that a black hole radiates. This phenomenon is called Hawking radiation. We derive the Hawking temperature for a massless scalar field in 2-dimensional spacetime.

3.5.1 Schwarzschild metric

A 4-dimensional non-rotating black hole with zero electric charge and mass M is described by the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 \left(d\theta^2 + d\phi^2 sin^2\theta \right) \tag{3.78}$$

We are using natural units here. For simplifying the calculations we consider a 2-dimensional black hole,

$$ds^2 = g_{ab} dx^a dx^b = \left(1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} \tag{3.79}$$

Introduce the tortoise coordinates

$$dr^* = \frac{dr}{1 - \frac{2M}{r}} \Rightarrow r^*(r) = r - 2M + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{3.80}$$

The metric then becomes

$$ds^2 = \left(1 - \frac{2M}{r(r*)}\right)[dt^2 - dr^{*2}] \tag{3.81}$$

The tortoise coordinate is only defined for $r > r_g$ and varies in the range $-\infty < r^* < +\infty$. Introduce the tortoise lightcone coordinates

$$\tilde{u} \equiv t - r^*, \quad \tilde{v} \equiv t + r^* \tag{3.82}$$

and write the metric as

$$ds^2 = \left(1 - \frac{2M}{r(\tilde{u}, \tilde{v})}\right) d\tilde{u}d\tilde{v} \tag{3.83}$$

These coordinates do not remove the singularity present at $r = 2M$ and do not cover the complete spacetime. They cover only the exterior of the black hole.

3.5.2 Kruskal-Szekeres coordinates

These coordinates describe the entire spacetime (apart from the singularity at $r = 0$). The Kruskal-Szekeres lightcone coordinates are defined as

$$u = -4Mexp\left(-\frac{\tilde{u}}{4M}\right), \quad v = 4Mexp\left(\frac{\tilde{v}}{4M}\right) \tag{3.84}$$

in which the metric takes the form

$$ds^2 = \frac{2M}{r(u,v)} exp\left(1 - \frac{r(u,v)}{2M}\right) dudv \tag{3.85}$$

The metric is now regular at $r = 2M$. This singularity is a coordinate singularity which can be removed by coordinate transformation. Also, as defined above, the Kruskal-Szekeres coordinates vary in the intervals $-\infty < u < 0$ and $0 < v < +\infty$, covering only the exterior of the black hole. However, they can be analytically extended to $u > 0$ and $v < 0$ so the Kruskal-Szekeres coordinates span the entire spacetime.We may find the original Schwarzschild coordinates t and r in terms of the Kruskal-Szekeres coordinates, if we consider these equations

$$uv = -16M^2 exp\left(\frac{r^*}{2M}\right) = -16M^2\left(\frac{r}{2M} - 1\right) exp\left(\frac{r}{2M} - 1\right) \tag{3.86}$$

and

$$\left(\frac{v}{u}\right)^2 = exp\left(\frac{2t}{2M}\right) \tag{3.87}$$

These equations are valid even for arbitrary u and v. We observe from these equations that the black hole horizon $r = 2M$ corresponds to $u = 0, v = 0$. $v = 0$ corresponds to $t = -\infty$ and $u = 0$ corresponds to $t = +\infty$. Thus, the Schwarzschild spacetime has two horizons, $v = 0$ the past horizon and $u = 0$ the future horizon.

3.5.3 Hawking temperature

Let us consider a massless scalar field with the action given by

$$S[\phi] = \frac{1}{2}\int g^{ab}\phi_{,a}\phi_{,b}\sqrt{-g}d^2x \tag{3.88}$$

in a 2-dimensional spacetime. To make our analysis easier, let us point out the mathematical similarities between the Minkowski/Rindler coordinates(for an accelerating observer) and the tortoise/Kruskal-Szekers coordinates(for a Schwarzschild black hole). The transformation between the corresponding lightcone coordinates is

$$u = -a^{-1}exp(-a\tilde{u}) \quad \text{and} \quad v = a^{-1}exp(a\tilde{v}) \quad \text{for Minkowski/Rindler}$$
$$u = -\kappa^{-1}exp(-\kappa\tilde{u}) \quad \text{and} \quad v = \kappa^{-1}exp(\kappa\tilde{v}) \quad \text{for Kruskal/tortoise} \tag{3.89}$$

where $\kappa = (2M)^{-1}$ is known as the surface gravity of the horizon. Also, Kruskal-Szekeres coordinates cover the entire spacetime, like Minkowski coordinates and the tortoise coordinates, like the Rindler coordinates, cover only the region exterior to the horizon.

The conformal invariant action allows us to write the solution of the scalar field equation in terms of tortoise lightcone coordinates as

$$\phi = \tilde{A}(\tilde{u}) + \tilde{B}(\tilde{v}) \tag{3.90}$$

or in terms of Kruskal-Szekeres lightcone coordinates as

$$\phi = A(u) + B(v) \tag{3.91}$$

where $A, \tilde{A}$,etc. are arbitrary smooth functions. As we did in the case of the scalar field in the Rindler spacetime, we can expand the quantized massless scalar field into its modes in the tortoise lightcone coordinates as

$$\hat{\phi} = \int \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega\tilde{u}} \hat{b}_{\Omega}^{-} + e^{i\Omega\tilde{u}} \hat{b}_{\Omega}^{+}] + (\text{left-moving modes}) \tag{3.92}$$

These are the right-moving modes w.r.t. time t, which move away from the black hole. The proper time of an observer at rest located at asymptotic infinity (far away from the black hole) coincides with t since $ds^2 \to d\tilde{u}d\tilde{v} = dt^2 - dr^{*2}$ as $r \to \infty$. This observer has the notion of particles w.r.t. the positive frequency modes w.r.t. time t. The creation and annihilation operators are $\hat{b}_{\Omega}^{\pm}$. The vacuum state corresponding to these operators is $|0_B\rangle$ where

$$\hat{b}_{\Omega}^{-} |0_B\rangle = 0 \tag{3.93}$$

$|0_B\rangle$ is called the Boulware vacuum. It has no particles from the point of view of an asymptotic observer. Boulware vacuum is similar to the Rindler vacuum.

Similarly, we can expand the field operator in Kruskal-Szekeres lightcone coordinates

$$\hat{\phi} = \int \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega u} \hat{a}_{\omega}^{-} + e^{i\omega u} \hat{a}_{\omega}^{+}] + (\text{left-moving modes}) \tag{3.94}$$

We define another set of creation and annihilation operator $\hat{a}_{\omega}^{\pm}$ that defines the Kruskal vacuum $|0_K\rangle$ as

$$\hat{a}_{\omega}^{-} |0_K\rangle = 0 \tag{3.95}$$

The Kruskal vacuum is analogous to Minkowski vacuum.

We can derive the occupation number in the same way as done in the previous section. The remote observer sees particles in the Kruskal vacuum with the thermal spectrum

$$\left\langle \hat{N}_{\Omega} \right\rangle \equiv \langle 0_K | \hat{b}_{\Omega}^{+} \hat{b}_{\Omega}^{-} | 0_K \rangle = \left[exp\left(\frac{2\pi\Omega}{\kappa} \right) - 1 \right]^{-1} \delta(0) \tag{3.96}$$

corresponding to the Hawking temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \tag{3.97}$$

Chapter 4

Anomalies and Hawking flux

4.1 What are anomalies?

Mostly, a symmetry of the classical theory is also a symmetry of the quantum theory based on the same Lagrangian. When that is not the case, the symmetry is said to be anomalous. Using Noether's theorem, we can find conserved currents for continuous global symmetries. If a symmetry is anomalous then it is not actually a symmetry and the associated current will not be conserved. For example, a gauge anomaly invalidates the gauge symmetry of a quantum field theory. A gravitational anomaly is a gauge anomaly which violates the general covariance of general relativity.

The author is further studying the precise analytical formulation of anomalies in quantum field theory.

4.2 Hawking flux using Anomalies

The discussion in this chapter follows [5] and the references cited therein.

4.2.1 Gravitational anomaly and basic setup

The previous derivation of Hawking radiation is based on calculating the Bogoliubov coefficients. Another approach to calculate the Hawking flux is to calculate the the energy-momentum tensor in the blackhole backgrounds. Classically, the EM tensor is covariantly conserved in a curved background. However, in quantum theory, the EM tensor is not necessarily conserved. For example, for a chiral scalar field in (1+1)-dimensional curved spacetime the covariant derivative of the EM tensor becomes

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{96\pi\sqrt{-g}}\epsilon^{\beta\delta}\partial_\delta\partial_\alpha\Gamma^\alpha{}_{\nu\beta} \tag{4.1}$$

The right hand side is the consistent gravitational anomaly([6][7][8][9]). It was shown by Christensen and Fulling [10], that under certain simplifying assumptions the above anomaly can be interpreted as a flux of radiation, which agrees with the Hawking flux quantitatively.

The above idea was shown to be valid for a variety of spacetimes by Robinson and Wilczek([11]). The basic idea in ([11]) is that the effective theory near the horizon becomes two-dimensional and chiral. With dimensional reduction procedure, we can effectively describe a theory with a metric given only by the "r-t" sector of the full spacetime metric. This chiral theory is anomalous. The two-dimensional covariant gravitational anomaly is used to calculate the Hawking flux. The metric used for our analysis is

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega \tag{4.2}$$

which can describe a variety of spacetimes like Schwarzschild, Reissner-Nordstrom, Kerr.

The boundary condition is obtained from a vanishing of energy-momentum tensor at the horizon.

4.2.2 Anomaly equations and solving for EM tensor

We divide the spacetime into two regions. In the region outside the horizon, the theory is anomaly-free and EM tensor remain conserved.

$$\nabla_\mu T^\mu_{(o)\nu} = 0 \tag{4.3}$$

Near the horizon in the region $r \in [r_+, \infty]$, the ingoing modes are lost to the black hole which leads to an anomaly in the EM tensor there. We take the covariant form of d=2 gravitational anomaly ([11][12])

$$\nabla_\mu T^\mu_{(H)\nu} = \frac{1}{96\pi}\bar{\epsilon}_{\nu\mu}\partial^\mu R = \mathcal{A}_\nu \tag{4.4}$$

where $\bar{\epsilon}^{\mu\nu} = \epsilon^{\mu\nu}/\sqrt{-g}$ and $\bar{\epsilon}_{\mu\nu} = \sqrt{-g}\epsilon_{\mu\nu}$ are two-dimensional antisymmetric tensors with $\epsilon^{tr} = \epsilon_{rt} = 1$.

Calculating both component of $\mathcal{A}$, we find that the anomaly is purely timelike

$$\mathcal{A}_r = 0 \qquad \mathcal{A}_t = \frac{1}{\sqrt{-g}}\partial_r N^r_t \tag{4.5}$$

where

$$N^r_t(r) = \frac{1}{96\pi}\left(hf'' + \frac{f'h'}{2} - \frac{f'^2 h}{f}\right) \tag{4.6}$$

Now we solve both the conservation and anomalous equations. Outside the horizon, the conservation equation yields,

$$\partial_r\left(\sqrt{-g}T^r_{(o)t}\right) = 0 \Longrightarrow T^r_{(o)t}(r) = \frac{a_o}{\sqrt{-g}} \tag{4.7}$$

where a_o is an integration constant.
Near the horizon, the anomaly equation leads to,

$$\partial_r\left(\sqrt{-g}T^r_{(H)t}\right) = \partial_r N^r_t(r) \Longrightarrow T^r_{(H)t} = \frac{1}{\sqrt{-g}}\left(b_H + N^r_t(r) - N^r_t(r_H)\right) \tag{4.8}$$

where b_H is an integration constant.

4.2.3 Hawking flux

As in [12][13], we can write the EM tensor as a sum of two contributions:

$$T^r{}_t(r) = T^r_{(o)t}(r)\theta(r - r_H - \epsilon) + T^r_{(H)t}(r)H(r) \tag{4.9}$$

where $H(r) = 1 - \theta(r - r_H - \epsilon)$. and $\theta(x)$ is the step function.
Calculating $\nabla_\mu T^\mu{}_t$, we find that

$$\begin{aligned}\nabla_\mu T^\mu{}_t &= \partial_r T^r{}_t(r) + \partial_r\left(ln\sqrt{-g}\right)T^r{}_t(r) = \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}T^r{}_t(r)\right)\\ &= \frac{1}{\sqrt{-g}}\left[\left(\sqrt{-g}\left(T^r_{(o)t}(r) - T^r_{(H)t}(r)\right) + N^r_t(r)\right)\delta(r - r_+ - \epsilon) + \partial_r(N^r_t(r)H(r))\right]\end{aligned} \tag{4.10}$$

The total derivative term is canceled by quantum effects of classically irrelevant ingoing modes. The vanishing of the Ward identity under diffeomorphism transformation means that the coefficient of the delta function in the above equation vanishes

$$T^r_{(o)t} - T^r_{(H)t} + \frac{N^r_t(r)}{\sqrt{-g}} = 0 \tag{4.11}$$

Substituting the solutions for $T^r_{(o)t}$ and $T^r_{(H)t}$ in the above equation, we have,

$$a_o = b_H - N^r_t(r_H) \tag{4.12}$$

Now we fix the integration constant b_H imposing the boundary condition, i.e., the vanishing of the covariant energy-momentum tensor at the horizon. We find $b_H = 0$. Hence, the total flux of the energy-momentum tensor is given by

$$a_o = -N^r_t(r_H) = \frac{1}{192\pi} f'(r_H) h'(r_H) \tag{4.13}$$

For Schwarzschild metric $f(r) = h(r) = (1 - \frac{2M}{r})$. Using this we obtain

$$a_o = \frac{1}{192\pi} \frac{1}{4M^2} \tag{4.14}$$

A beam of massless black body radiation moving outwards in the radial direction at a temperature T_H has a flux of the form:

$$a_o = \frac{\pi}{12} T_H^2 \tag{4.15}$$

Comparing we get,

$$T_H = \frac{1}{8\pi M} \tag{4.16}$$

which is in complete agreement with the Hawking temperature obtained in the previous chapter.

Chapter 5

Conclusion

In this thesis work, we reviewed the basic concepts of general relativity and quantum field theory in curved spacetime and henceforth graduated to quantum field theory in curved spacetime. As a first order approximation to the full theory of quantum gravity, quantum field theory in curved spacetime gave us some interesting and unexpected results such as Unruh effect and Hawking radiation. The main difference between flat and curved spacetime is that in curved spacetime, once cannot define a unique vacuum state. We calculate the Unruh and Hawking temperature using Bogoliubov coefficients.

An anomaly is the breaking of a classical symmetry at the quantum level. Anomalies were introduced in a qualitative fashion. The gravitational anomaly was shown successfully to yield the Hawking temperature. Further study is being done by the author in the area of anomaly and generalisation of the above method for higher spin fields.

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