Analysis of Table Tennis Singles' Matchups

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A Comprehensive Study of General and Special Cases
Default Rules as in the `webpage`*.*

General Case: Probabilistic Framework Consider two teams, A and B, each with N players: $A_1, A_2, \ldots, A_N$ for Team A and $B_1, B_2, \ldots, B_N$ for Team B. The players are ranked such that $A_1 > A_2 > \cdots > A_N$ and $B_1 > B_2 > \cdots > B_N$, where $>$ denotes a stronger ranking. In a table tennis singles match, each player A_i from Team A faces one player $B_{\sigma(i)}$ from Team B, where σ is a permutation of $\{1, 2, \ldots, N\}$, chosen uniformly at random from the $N!$ possible permutations.

Define p_{ij} as the probability that A_i defeats B_j. For each matchup A_i versus $B_{\sigma(i)}$, let the indicator variable X_i represent the outcome:

$$X_i = \begin{cases} 1, & \text{if } A_i \text{ beats } B_{\sigma(i)}, \\ 0, & \text{otherwise.} \end{cases}$$

The total number of games won by Team A is:

$$S_A = \sum_{i=1}^{N} X_i.$$

Since σ is random, the expected number of wins for A_i is:

$$E[X_i] = \sum_{j=1}^{N} P(\sigma(i) = j) \cdot p_{ij} = \frac{1}{N} \sum_{j=1}^{N} p_{ij},$$

because each B_j is equally likely to be assigned to A_i with probability $\frac{1}{N}$. Thus, the expected total wins for Team A are:

$$E[S_A] = \sum_{i=1}^{N} E[X_i] = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij}.$$

The probability that Team A wins exactly k games, $P(S_A = k)$, is:

$$P(S_A = k) = \frac{\text{Number of permutations } \sigma \text{ where } \sum_{i=1}^{N} X_i = k}{N!}.$$

To determine the match outcome:

- Team A wins if $S_A > N/2$ (more than half the games).
- Tie if $S_A = N/2$ (possible only if N is even).
- Team B wins if $S_A < N/2$.

Thus:

$$P(\text{Team A wins}) = \sum_{k>N/2} P(S_A = k), \tag{1}$$

$$P(\text{Tie}) = P(S_A = N/2) \quad (\text{if } N \text{ is even, else } 0), \tag{2}$$

$$P(\text{Team B wins}) = \sum_{k<N/2} P(S_A = k). \tag{3}$$

For large N, S_A approximates a normal distribution via the Central Limit Theorem, with mean $E[S_A]$ and variance based on the X_i's, but we focus on exact computation for the special case.

Special Case: Deterministic Outcomes with $N = 4$ Now, consider a specific instance where $N = 4$, and the win conditions are deterministic ($p_{ij} = 0$ or 1):

- A_1 beats B_2, B_3, B_4, loses to B_1.
- A_2 beats B_2, B_3, B_4, loses to B_1.
- A_3 beats B_3, B_4, loses to B_1, B_2.
- A_4 beats B_4, loses to B_1, B_2, B_3.

The win probability matrix $P = (p_{ij})$ is:

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where rows correspond to A_1, A_2, A_3, A_4 and columns to B_1, B_2, B_3, B_4.

There are $4! = 24$ equally likely permutations. For each σ, compute $S_A = \sum_{i=1}^{4} p_{i,\sigma(i)}$. The possible values of S_A range from 0 to 4: $P(S_A = 4) = 0$, $P(S_A = 3) = \frac{8}{24} = \frac{1}{3}$, $P(S_A = 2) = \frac{14}{24} = \frac{7}{12}$, $P(S_A = 1) = \frac{2}{24} = \frac{1}{12}$, $P(S_A = 0) = 0$.

Final Probabilities

- $P(\text{Team A wins}) = P(S_A \geq 3) = P(S_A = 3) = \frac{1}{3}$,
- $P(\text{Tie}) = P(S_A = 2) = \frac{7}{12}$,
- $P(\text{Team B wins}) = P(S_A \leq 1) = P(S_A = 1) = \frac{1}{12}$.