A Differentiable Monster

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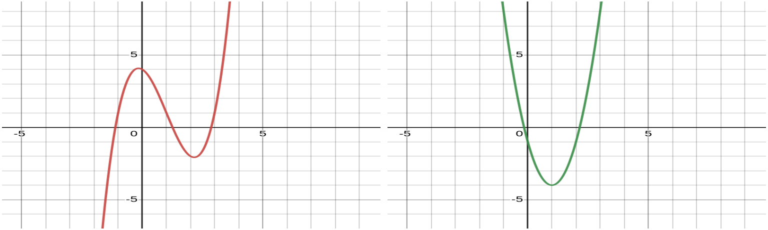
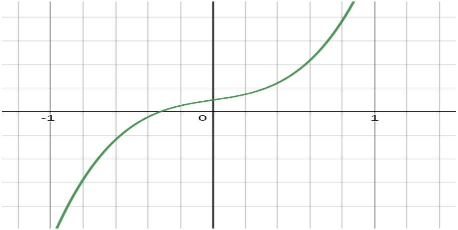


Figure: A differentiable function[left] and its derivative[right]

Let $(a, b) \subset \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be differentiable. Then:

- f is continuous on (a, b).
- f' follows the intermediate value property.

Monotonicity



A function $f : \mathbb{R} \to \mathbb{R}$ is **increasing** on $(a, b) \subset \mathbb{R}$ if

$$f(x) \leq f(y)$$

for $x, y \in (a, b)$ and $x < y$.
Similarly, it is **decreasing** if $f(x) \geq f(y)$.

We say a function is **monotone** on (a, b) if it is either increasing or decreasing on (a, b).

Not monotone

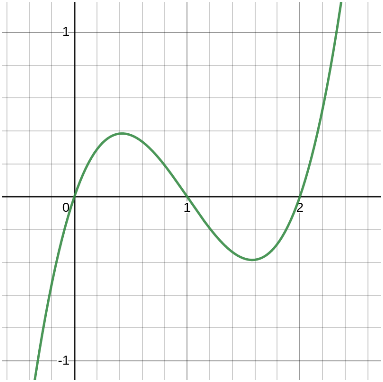
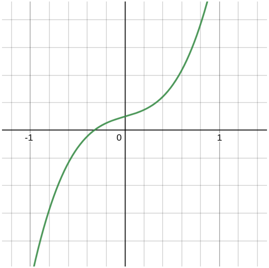


Figure: A function that is not monotone on $(0, 1)$

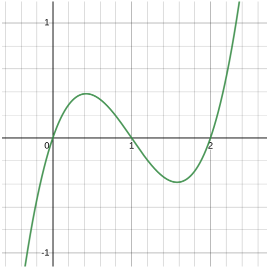
Monotonicity and differentiable functions



If a differentiable function is increasing in an interval, then $f' \geq 0$ in that interval.

Sign of derivative and monotonicity

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is continuous ($\mathcal{C}^1$ function).
If $f'(a) > 0$, then f is increasing in an interval of 'a'.



What if we drop continuity of derivative?

If a differentiable function f be such that $f'(0) > 0$, then does there exists a interval about 0 where f is monotone?

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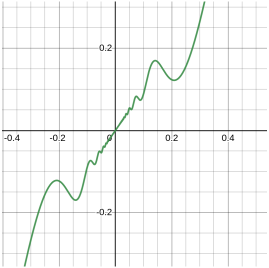
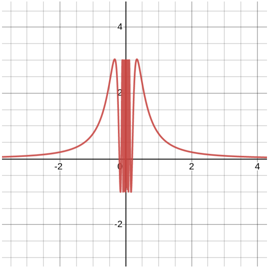
No!





$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x + 2x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

Nowhere monotone?

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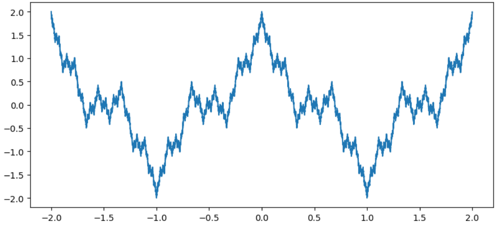

Figure: Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

Don’t these functions feel very “rugged”?

How “smooth” are differentiable functions

We have seen that nowhere monotone functions are very **rugged** and we have a notion of **“smoothness”** (*no sharp edges*) associated to differentiable functions.

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Yes!

Summoning a monster

There exists a nowhere monotone differentiable function.

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AIM

We want to find a function f such each of the sets

$$\{x \in \mathbb{R} : f'(x) > 0\} \text{ and } \{x \in \mathbb{R} : f'(x) < 0\}$$

is dense sets in $\mathbb{R}$. Then f cannot be monotone in any non-empty open interval.

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- Given any open interval: say (a, b)
- If f were increasing in (a, b), the $f'(x) \geq 0$ for $x \in (a, b)$.
- Since the set of points where f' is negative is dense in $\mathbb{R}$. There would be a point in (a, b) with derivative negative.

Differentiable Monster

A **differentiable monster**
is a nowhere monotone differentiable function.

Theorem (Alfred Köpcke - 1887)

There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere monotone. In particular, each of the sets

$$Z_f := \{x \in \mathbb{R} : f'(x) = 0\} \text{ and } Z_f^c := \{x \in \mathbb{R} : f'(x) \neq 0\}$$

is dense in $\mathbb{R}$, and f' is discontinuous at each point of Z_f^c.

G_δ sets

Definition (G_δ set)

A G_δ set is a countable intersection of open sets.

Examples

The zero set of a continuous function is a (closed) G_δ set

$$f^{-1}(0) = \bigcap_{n>0} f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

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Baire's Theorem

Countable intersection of dense open sets in $\mathbb{R}$ is a dense G_δ set.

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Very Large!

Lemma (Pointwise limit of continuous functions)

Let $g(x) = \lim_{n\to\infty} g_n(x)$ where $g_n : \mathbb{R} \to \mathbb{R}$ is continuous.
Then, the set of points of continuity of g is a dense G_δ set.

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Theorem (Continuity of derivative)

Let $F : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
Then, the set of points of continuity of F' is a dense G_δ set.

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Theorem (Continuity of derivative)

Let $F : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
Then, the set of points of continuity of F' is a dense G_δ set.

Proof

$$g_n(x) := \frac{F(x + 1/n) - F(x)}{1/n}$$

By the definition of differentiation $lim_{n\to\infty} g_n(x) = F'(x)$. We are done by the above lemma.

Pompeiu's Function

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Pompeiu's Function

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Proposition (Dimitrie Pompeiu - 1907)

There exists a strictly increasing differentiable function $h : \mathbb{R} \to \mathbb{R}$ for which each of the sets

$$Z_h := \{x \in \mathbb{R} : h'(x) = 0\} \text{ and } Z_h^c := \{x \in \mathbb{R} : h'(x) > 0\}$$

is dense in $\mathbb{R}$.

Such a function is called as a **Pompeiu's function**.

Intuition for Pompeiu's Function

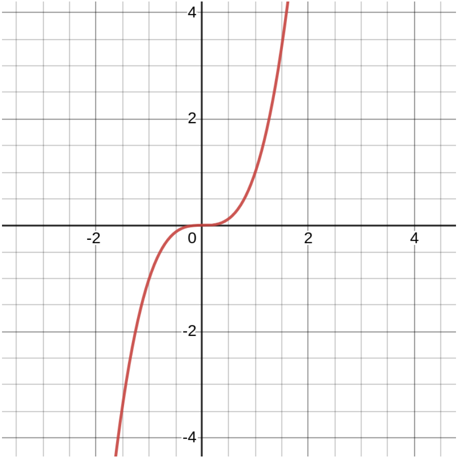

Figure: $f(x) = x^3$

Intuition for Pompeiu's Function (Cont.)

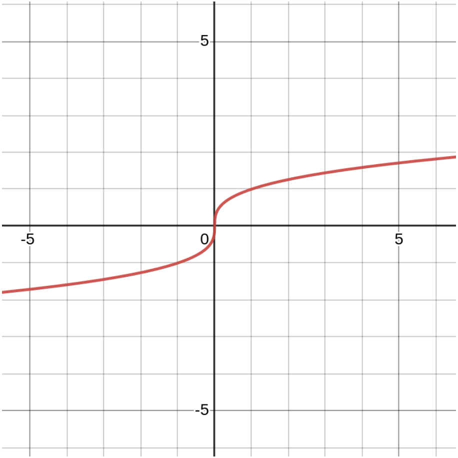

Figure: $f(x) = x^{\frac{1}{3}}$

Intuition for Pompeiu's Function (Cont.)

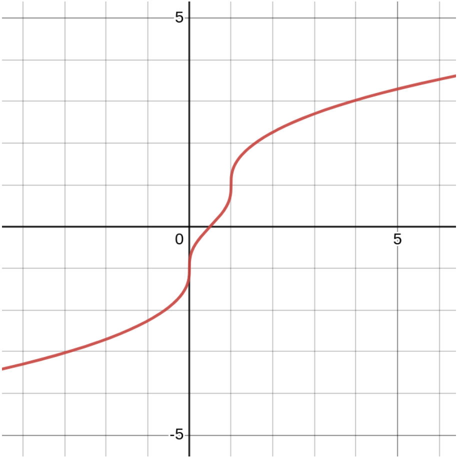

Figure: $f(x) = x^{\frac{1}{3}} + (x-1)^{\frac{1}{3}}$

Intuition for Pompeiu's Function (Cont.)

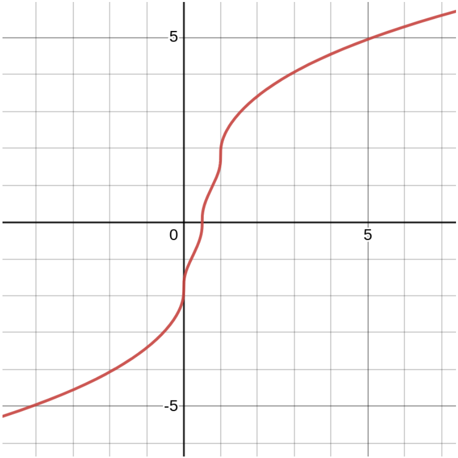

Figure: $f(x) = x^{\frac{1}{3}} + (x - \frac{1}{2})^{\frac{1}{3}} + (x - 1)^{\frac{1}{3}}$

Intuition for Pompeiu's Function (Cont.)

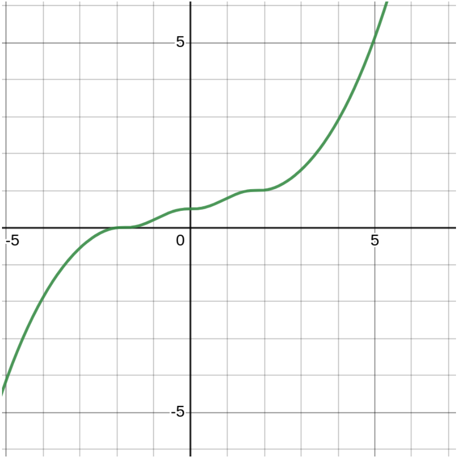

Figure: $x = f(y)$

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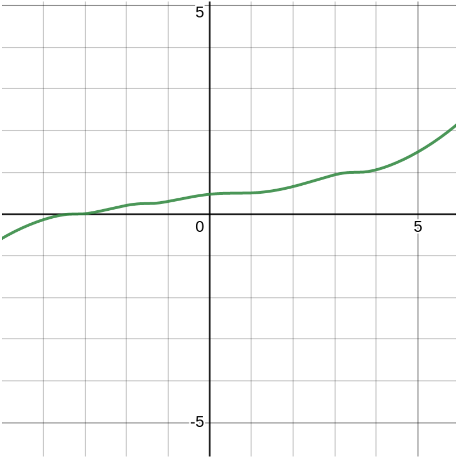

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A Pompeiu's Function h

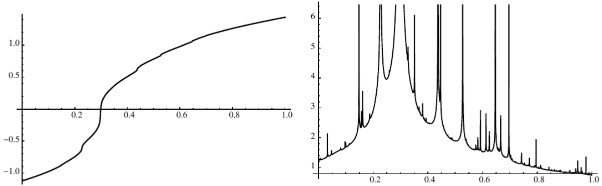

Figure: g[left] and g'[right]

Take an enumeration $\mathcal{Q} := \{q_j : j \in \mathbb{N}\}$ of $\mathbb{Q}$, such that $|q_j| \leq j$. Let $0 < r < 1$.

$$g(x) := \sum_{j=1}^{\infty} r^j (x - q_j)^{\frac{1}{3}}$$

Then g is continuous and strictly increasing.
It has an inverse h which is a strictly increasing differentiable function with h' being zero on $g(\mathbb{Q})$.

Continuity of h'

We denoted $Z_h = \{x \in \mathbb{R} : h'(x) = 0\}$ and $Z_h^c = \{x \in \mathbb{R} : h'(x) > 0\}$. They are both dense in $\mathbb{R}$.

- h' can't be continuous on Z_h^c as Z_h is dense in $\mathbb{R}$. So the set of points of continuity of h' must be in Z_h^c
- Z_h contains a dense G_δ set.

Z_h is a very large set

Monster from Pompeiu's function

Let h be a Pompeiu's function. Then:

- $Z_h = \{x \in \mathbb{R} : h'(x) = 0\}$ contains the points of continuity of h' so it contains a dense G_δ set. (This is a very large set)
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  - $G := \bigcap_{\eta \in D} ((-\eta + Z_h) \cap (\eta - Z_h))$
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it's here, run!

Then $f(x) := h(x - t) - h(x)$ is a differentiable monster!

Examining the monster

- $Z_h = \{x \in \mathbb{R} : h'(x) = 0\}$ and $Z_h^c = \{x \in \mathbb{R} : h'(x) > 0\}$
- Let t be such that $D + t \subset Z_h$ and $D - t \subset Z_h$.
- $f(x) := h(x - t) - h(x)$
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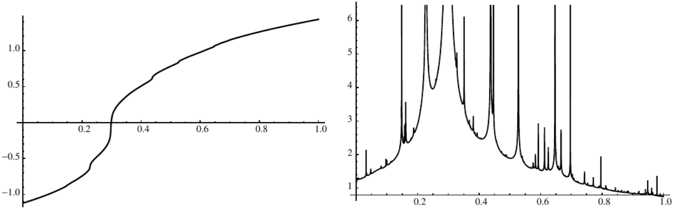
f cannot be monotone in any non-empty open set!

Pompeiu's Function in Depth



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- Since each term is strictly increasing, g is strictly increasing.

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- g uniformly converges on compact sets. Hence g is continuous.
- Since each term is strictly increasing, g is strictly increasing.
- In this setting

$$g'(x) = \sum_{j=1}^{\infty} r^j \frac{1}{3(x - q_j)^{\frac{2}{3}}}$$

when the sum converges. Otherwise we have $g'(x) = +\infty$. In particular g' is $+\infty$ for points in $\mathcal{Q}$.

A Pompeiu's Function(Cont.)

- As g is strictly increasing and continuous, it has a inverse h which is strictly increasing and continuous.

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- As g is strictly increasing and continuous, it has a inverse h which is strictly increasing and continuous.
- h' is zero on $g(\mathbb{Q})$. And h' is finite on every point as $g' > 0$ for every point.
- $h : \mathbb{R} \to \mathbb{R}$ is differentiable and Z is dense:

$$g(\mathbb{Q}) \subset Z = \{x \in \mathbb{R} : h'(x) = 0\}$$

- Since h is strictly increasing

$$Z^c = \{x \in \mathbb{R} : h'(x) > 0\}$$

is dense.

References

- Bull. Amer. Math. Soc. **56** (2019), 211-260 : Differentiability versus continuity: Restriction and extension theorems and monstrous examples
- Continuous Nowhere Differentiable Functions: The Monsters of Analysis - Marek Jarnicki and Peter Pflug

Hope you were not scared by the monster.

Thank you!