NEWTON'S LAWS - THE GEOMETRIC FORMULATION
(My understanding so far...)

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Definition. A Newtonian Spacetime is a quintuple of structures $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$ where $(M, \mathcal{O}, \mathcal{A})$ is a 4-dimensional smooth manifold and $t : M \to \mathbb{R}$ satisfying the following:

> It's a set equipped with a topology which is smooth (so that we can talk about derivatives) and a covariant derivative (because Newton talks of uniform-straight lines which become just straight lines in ***spacetime***) and something called ***absolute time***.
> Don't think of ***spacetime*** as ***space*** + ***time***. Put it on an equal footing just like there is no preferred direction in S^2.

1. There is an ***absolute space***.
   Meaning $(dt)_p \neq 0 \, \forall \, p \in M$.

   > **Definition.** ***Absolute space*** at time τ is $S_\tau := \{p \in M \,|\, t(p) = \tau\}$
   > Then $(dt)_p \neq 0 \Rightarrow M = \dot{\bigcup} S_\tau$

2. $\boldsymbol{\nabla}$ is ***torsion-free***.
   So that $\Gamma^k_{ji} = \Gamma^k_{ij}$

3. ***Absolute time*** flows uniformly. (compatibility axiom)
   Meaning $\boldsymbol{\nabla} dt = 0$ everywhere.

   > Equality in the space of (0,2) tensor fields.

Definition. A vector $X \in T_pM$ is called

1. Future-directed if
   $dt(X) > 0$
2. Spatial (in the sense of absolute space) if
   $dt(X) = 0$
3. Past-directed if
   $dt(X) < 0$

NEWTON'S I LAW : The worldline of a particle under the influence of no force (gravity isn't one anyway) is a ***future-directed autoparallel.***

> Autoparallel : $\nabla_{v_x} v_x = 0$ (no acceleration / no force)
> Future-directed : $dt(v_x) > 0$

NEWTON'S II LAW : $\nabla_{v_x} v_x = \dfrac{F}{m} \Leftrightarrow ma = F$

> where remember a is the complete acceleration.
> X is future-directed.
> And F is a spatial vector field $\Rightarrow dt(F) = 0$.

Now one follows a convention of restricting attention to atlases, $\mathcal{A}_{\text{stratified}}$ whose charts $(\mathcal{U}, x)$ have the property

$$x^0 : \mathcal{U} \to \mathbb{R} \text{ such that } x^0 = t\,|_{\mathcal{U}}$$
$$x^1 : \mathcal{U} \to \mathbb{R}$$
$$x^2 : \mathcal{U} \to \mathbb{R}$$
$$x^3 : \mathcal{U} \to \mathbb{R}$$

so that only the formulae that one derives later become simple. Other than this there is no other motive for one to do this.

Now going back to the assumptions of Newton's Laws, one finds, Time flows uniformly

$$\begin{aligned} &\Rightarrow \nabla dt = 0 \\ &\Rightarrow \nabla_{\frac{\partial}{\partial x^a}} dx^0 = 0 \qquad a = 0, 1, 2, 3 \\ &\Rightarrow -\Gamma^0_{ba} = 0 \end{aligned}$$

Hence, in a stratified atlas, $\Gamma^0_{ab} = 0 \,\forall\, a, b$.

Now, evaluating in a chart $(\mathcal{U}, x)$ of a stratified atlas, $\mathcal{A}_{\text{stratified}}$, Newton's II Law tells us:

$$\nabla_{v_x} v_x = \frac{F}{m} \text{ (in some parametrisation } \lambda)$$

In a chart,

$$\ddot{X}^0 + \Gamma^0_{cd} \dot{X}^c \dot{X}^d = \frac{F^0}{m} = \frac{dt(F)}{m} = 0$$
$$\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + \Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0 = \frac{F^\alpha}{m}$$

Hence, in a stratified atlas,

$$\begin{aligned} &\ddot{X}^0(\lambda) = 0 \\ &\Rightarrow X^0(\lambda) = a\lambda + b \qquad a, b \in \mathbb{R} \\ &\Rightarrow (x^0 \cdot X)(\lambda) = a\lambda + b \\ &\Rightarrow (t \cdot X)(\lambda) = a\lambda + b \end{aligned}$$

One follows a convention of parametrising the worldline by absolute time,

$$\Rightarrow \frac{d}{d\lambda} = a\frac{d}{dt}$$

Hence, Newton's II Law in a chart becomes,

$$a^2\ddot{X}^\alpha + a^2\Gamma^\alpha_{\gamma\delta}\dot{X}^\gamma\dot{X}^\delta + a^2\Gamma^\alpha_{00}\dot{X}^0\dot{X}^0 + 2a^2\Gamma^\alpha_{\gamma 0}\dot{X}^\gamma\dot{X}^0 = \frac{F^\alpha}{m}$$

where a^α is the LHS. The whole of it is acceleration. Not just one or a combination of terms in the LHS. Because remember, any one term goes missing, it does not transform like a vector. But acceleration is a vector. Which implies that it's a tensor and if it has a vanishing component in one chart, it has vanishing component in any chart. But now one has to bear in mind that it's a chart in the spacetime!
If one wants a rotating co-ordinate system, one can just twist the same chart with evolving time. Hence one chart in spacetime is a rotating chart in space! Isn't that wonderful!!
Due to gravity, we might have the term Γ^α_{00} which cannot be transformed away. If there's no gravity, there is a co-ordinate system where all Γ's vanish, leading to

$$\ddot{X}^\alpha = \frac{1}{a^2}\cdot\frac{F^\alpha}{m} \qquad \Rightarrow \text{Newtonian Spacetime is FLAT!}$$
$$\alpha = 0, 1, 2, 3$$

But if one tries to think of a rotating frame, where in the chart is being twisted in spacetime, all sorts of Γ's popup!

$\Gamma^{\alpha}_{\gamma\delta}\dot{X}^{\gamma}\dot{X}^{\delta}$ are the correction terms that come into picture when we choose a different co-ordinate system.
Γ^{α}_{00} comes into picture when we have a pseudo centrifugal acceleration.
$2\Gamma^{\alpha}_{\gamma 0}\dot{X}^{\gamma}$ implies that there is a pseudo coriolis force.

Pseudo because we're not certain and that's right because all the terms put together make up for the acceleration. Not just one or two terms.

So according to Newton's I Law, it just doesn't matter which co-ordinate system you are in, the complete a is zero. That's it.

Remember, how do you know you are in an inertial frame?

$$\nabla_{\frac{\partial}{\partial x^{\alpha}}}\frac{\partial}{\partial x^{\beta}} = 0$$

$\Rightarrow$ All the Γ's are zero!!