Ordinary Differential Equations
Part 2

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- Becomes a bigger problem when the solution to an ODE has both rapidly varying and slowly varying terms.

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  - $h < \frac{1}{15}$, then $0 < (1 - 15h) < 1$: Euler solution decreases monotonically to zero - like the exact solution!

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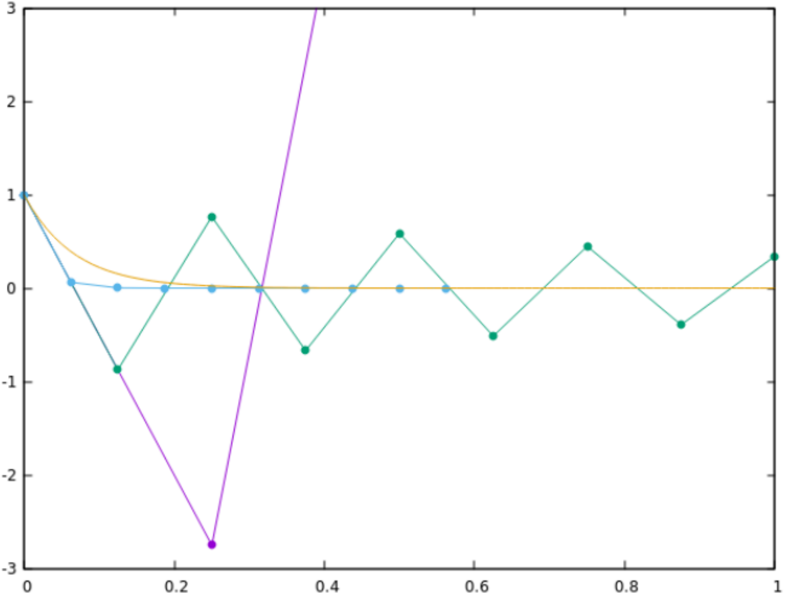

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- It is *unconditionally stable*!

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We have either the **Adams-Bashforth methods:**

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The former are explicit methods, while the latter are implicit.

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as

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Using backwards Taylor series, we have

$$\begin{aligned} & hy'\left(x_n\right) - \frac{h^2}{2} y''\left(x_n\right) + \frac{h^3}{3!} y'''\left(x_n\right) - \frac{h^4}{4!} y^{IV}\left(x_n\right) + \ldots \\ & - h\beta_0 y'\left(x_n\right) - h\beta_1 \left(y'\left(x_n\right) - hy''\left(x_n\right) + \frac{h^2}{2!} y'''\left(x_n\right)\right) + \ldots = 0 \end{aligned}$$

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C_1 &= 1 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 \ldots \\
C_2 &= -\frac{1}{2} + (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + \ldots) \\
C_3 &= +\frac{1}{6} - \frac{1}{2}(\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + \ldots) \\
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- This gives $\beta_1 = \dfrac{55}{24}$, $\beta_2 = -\dfrac{59}{24}$, $\beta_3 = \dfrac{37}{24}$, $\beta_4 = -\frac{3}{8}$.

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- For Adams-Moulton, we allow $\beta_0 \neq 0$.
- For $k = 1$, we can obtain a method accurate up to h^2 by chosing β_0 and β_1 so that $C_1 = C_2 = 0$.
- This leads to $\beta_0 = \beta_1 = \frac{1}{2}$
- For $k = 3$, we can get a solution accurate up to $h^4$by solving $C_1 = C_2 = C_3 = C_4 = 0$, alongh with $\beta_4 = 0$.
- This gives

$$\beta_0 = \frac{3}{8},\ \beta_1 = \frac{19}{24},\ \beta_2 = -\frac{5}{24}, \beta_3 = \frac{1}{24}$$

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- We follow this up with an Adams-Moulton step to correct the y_n:

$$y_n = y_{n+1} + h\beta_0 f\left(x_n, y_n^*\right) + h \sum_{i=1}^{k} \beta_i f\left(x_{n-i}, y_{n-i}\right)$$

- Typically we use $k^* = k + 1$

Using the Adams methods

- But ... how do we start?
- This approach requires k values $y_0, y_1, \ldots, y_{k-1}$ to start !
- One approach could be to use an appropriate order RK method to get these.
- There are other methods as well.