Lecture Notes on Electromagnetism

Abstract. The contents of this text is based on the class notes on Electromagnetism for the PH412 course by Prof. Ananda Dasgupta, IISER Kolkata.

Contents

CHAPTER 1

Introduction

1.1. Lecture 1 : Covariance & Lorentz Transformation

1.1.1. Meaning of Covariance. Consider the following Maxwell equation

$$\nabla.E = -\rho/\epsilon_0$$

and the transformation of the co-ordinates

$$x^\mu \rightarrow x^{'\mu}$$

The equation is said to be covariant under the given transformation if both sides of it vary in such a way, that in the transformed(primed) co-ordinate system, the equation again holds true i.e.,

$$\nabla^{'}.E^{'} = -\rho^{'}/\epsilon_0^{'}$$

Now consider the vector identity

$$\vec{A}X(\vec{B}X\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$$

If we are asked to verify this, one of the simple ways is to choose the vectors in a convenient way viz.

$$\begin{aligned}\vec{C} &= c\hat{i}\\ \vec{B} &= b_1\hat{i} + b_2\hat{j}\\ \vec{A} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\end{aligned}$$

It is then easy to see that both sides of the equation are equal. Now we can rotate the co-ordinate system in any way but this equation identity will still hold true due to it's covariance under rotation. Some equations are manifestly covariant eg., $A^\mu = B^\mu$. Since the components are equal they will change equally under any transformation.

1.1.2. The Lorentz Transformation. The distance between two points (x^μ and y^μ) in flat space-time is :

$$\begin{aligned}l^2 &= \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) \qquad (1.1)\\ \text{where, } \eta_{\mu\nu} &= diag(1, -1, -1, -1)\end{aligned}$$

A transformation of the co-ordinates $x^\mu \rightarrow x^{'\mu}$ & $y^\mu \rightarrow y^{'\mu}$ such that the distance l^2 is preserved is called a Lorentz transformation i.e., Lorentz transformation preserves the interval between any two events in space-time. As a special case, the distance between the origin and any space-time point x^μ $(= x^\mu x_\mu)$ is preserved by Lorentz transformation. Note that simple translations of the co-ordinate axes can also preserve space-time intervals. However we are not interested in such transformations.

An example of a simple Lorentz transformation is:

$$\begin{aligned} ct^{'} &\equiv x^{'0} = \gamma(ct - \beta x) \\ x^{'} &\equiv x^{'1} = \gamma(-\beta ct + x) \\ y^{'} &\equiv x^{'2} = y \\ z^{'} &\equiv x^{'3} = z \end{aligned}$$

It can be represented in the matrix form as:

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{1.2}$$

A Lorentz transformation can be written as

$$\begin{aligned} X^{'} &= LX \\ Let\ (L)_{\mu\nu} &= \Lambda^\mu{}_\nu \\ x^{'\mu} &= \Lambda^\mu{}_\nu x^\nu \end{aligned}$$

where the summation convention has been used. If an index appears twice in the same expression (even in the same variable), once upstairs and once downwards then a sum is carried over that index. Also note that the Greek indices (μ, ν, etc) take on values 0, 1, 2, 3 while the Latin indices (i, j, k, etc) take on the values 1, 2, 3. $x^\mu \equiv (x^0, x^i) \equiv (ct, x)$. Now,

$$\begin{aligned} x^\mu x_\mu &= x^{'\mu} x^{'}_\mu \\ \Longrightarrow X^T \eta X &= X^{'T} \eta X^{'} = (LX)^T \eta (LX) \\ \Longrightarrow X^T \eta X &= X^T (L^T \eta L) X \end{aligned}$$

Does this mean $L^T \eta L = \eta$? Yes it does and the key lies in the fact that η is symmetric.

Proof:

If we have,

$X^T C X = 0$, we could choose $X^\mu = \delta_{\mu\nu}$ to conclude $C_{\nu\nu} = 0$. We could also choose $X^\mu = \delta_{\mu\nu} + \delta_{\mu\rho}$ to conclude $C_{\rho\nu} = -C_{\nu\rho}$ ($\nu \neq \rho$). Thus $C^T = -C$ i.e., C is an anti-symmetric matrix.

Thus we can conclude

$$L^T \eta L = \eta + C$$

where C is any anti-symmetric matrix. If we take the transpose on both sides and use $\eta^T = \eta$, we get $C^T = C$, but $C^T = -C$. Thus $C = 0$. We get

$$L^T \eta L = \eta \tag{1.3}$$

If we take the determinant of the matrices on both sides, we get $det(L) = \pm 1$. It can be shown that $(\Lambda^0{}_0)^2 \geq 1$ [Exercise 1]. Based on the last two statements, there are four classes of Lorentz transformations viz.

(1) $|L| = 1, \Lambda^0{}_0 \geq 1$ denoted $L_+^\uparrow$ eg. the transformation matrix at the beginning of this section.
(2) $|L| = 1, \Lambda^0{}_0 \leq -1$ denoted $L_+^\downarrow$ eg. $P.T$ where P is the parity transformation and T is the time reversal transformation.
(3) $|L| = -1, \Lambda^0{}_0 \geq 1$ denoted $L_-^\uparrow$ eg. $P = diag(1, -1, -1, -1)$.
(4) $|L| = -1, \Lambda^0{}_0 \leq -1$ denoted $L_-^\downarrow$ eg. $T = diag(-1, 1, 1, 1)$.

It can be shown that Lorentz transformations (L) form a group with the use of the fact that L is a linear transformation such that $L^T \eta L = \eta$ [Exercise 2] It can also be shown that only $L_+^\uparrow$ forms a proper subgroup of the group of Lorentz transformations [Exercise 3]. This is known as the proper orthochronous Lorentz transformation.

Under Lorentz transformation, the co-ordinates transform as $X' = LX$. Any four-vector also transforms in the same way. However the product $\Sigma_\mu A^\mu B^\mu$ is not conserved under Lorentz transformation. In 3D, however, this does hold true since the transformation matrix, R (the rotation matrix) is orthogonal. Thus, in 3D

$$X'^T Y' = X^T Y$$

1.1.3. Summary. In summary the following are the properties of Lorentz transformations (L) :

- They preserve the interval between two space-time events.
- $L^T \eta L = \eta$ where $\eta = diag(1, -1, -1, -1)$. The set of Lorentz transformations form a group.
- $det(L) = \pm 1$, ${\Lambda^0}_0 \geq 1$ or ${\Lambda^0}_0 \leq 1$. Thus there are four classes of Lorentz transformations of which only $L_+^\uparrow$ is a proper subgroup of the Lorentz group.

1.1.4. List of Exercises.

(1) Show that $({\Lambda^0}_0)^2 \geq 1$
(2) Show that Lorentz transformations (L) form a group with the use of the fact that L is a linear transformation such that $L^T \eta L = \eta$
(3) Show that only $L_+^\uparrow$ forms a proper subgroup of the group of Lorentz transformations.

1.2. Lecture 2 : Tensors

1.2.1. Covariant and Contravariant. Quantities which transforms like the co-ordinate differentials under co-ordinate transformations are called contravariant vectors. Suppose under a transformation L, the co-ordinate vectors $\underline{X}$ transform as

$$\underline{X}' = L\underline{X} \tag{1.4}$$

This can also be written as

$$X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu} \tag{1.5}$$

where

$$\Lambda^{\mu}{}_{\nu} = L_{\mu\nu} \tag{1.6}$$

then the transformation of the differentials of the coordinates will be given by the usual relation[1],

$$dX'^{\mu} = \frac{\partial X'^{\mu}}{\partial X^{\nu}} dX^{\nu} \tag{1.7}$$

Now any quantity which transforms as

$$A'^{\mu} = \frac{\partial X'^{\mu}}{\partial X^{\nu}} A^{\nu} \tag{1.8}$$

is defined as a contravariant vector or simply a vector.
The quantity $\frac{\partial X'^{\mu}}{\partial X^{\nu}}$ can be re-written as

$$\begin{aligned} \frac{\partial X'^{\mu}}{\partial X^{\nu}} &= \partial_{\nu}(\Lambda^{\mu}{}_{\rho} X^{\rho}) \\ &= \Lambda^{\mu}{}_{\rho} \partial_{\nu} X^{\rho} \\ &= \Lambda^{\mu}{}_{\rho} \delta^{\rho}{}_{\nu} \\ &= \Lambda^{\mu}{}_{\nu} \end{aligned} \tag{1.9}$$

Now let us have a look at quantities like $\sum_{\mu=0}^{3} A^{\mu} B^{\mu}$.

$$\sum_{\mu=0}^{3} A'^{\mu} B'^{\mu} = \sum_{\mu=0}^{3} (\Lambda^{\mu}{}_{\nu} A^{\nu})(\Lambda^{\mu}{}_{\rho} B^{\rho})$$

In matrix notation this stands as

$$\begin{aligned} \underline{A}' &= L\underline{A} \\ \underline{B}' &= L\underline{B} \\ \underline{A}'^{T} \underline{B}' &= \underline{A}^{T} L^{T} L \underline{B} \end{aligned}$$

Evidently, it is not an invariant quantity, since $L^{T} L$ need not necessarily be $\mathbf{1}$.
Let us see, if we can make the quantity $\underline{A}' \underline{B}'$ invariant by choosing some other transformation rule for B.

$$\begin{aligned} \underline{A}' &= L\underline{A} \\ \underline{B}' &= M\underline{B} \\ \underline{A}'^{T} \underline{B}' &= \underline{A}^{T} L^{T} M \underline{B} \end{aligned}$$

---

[1] The differentials do transform linearly even under any arbitrary non-linear transformation $X'^{\mu} = X'^{\mu}(X^{\nu})$

By choosing

$$M = (L^T)^{-1} \tag{1.10}$$

we can ensure

$$\underline{A}'^T \underline{B}' = \underline{A}^T \underline{B}$$

The quantity B which transforms like

$$\underline{B}' = (L^T)^{-1}\underline{B}$$

is defined to be a covariant vector or a covector[2]. A covariant vector (covector) maps a contravariant vector (vector) linearly to a scalar. They can be thought of as dual vectors similar to bras and kets.

$$\underline{B}^T \underline{A} \longrightarrow \text{scalar} \tag{1.11}$$

An example of covariant vector is the gradient of a scalar, $\vec{\nabla}\phi$.

$$\vec{\nabla}\phi \cdot d\vec{r} = d\phi$$

$\vec{\nabla}\phi$ maps the vector $d\vec{r}$ to a scalar $d\phi$. We can explicitly check how $\vec{\nabla}\phi$ transforms.

$$\begin{aligned} \vec{\nabla}\phi = & \ \frac{\partial\phi}{\partial X^\mu} e^\mu \\ = & \ \partial_\mu \phi e^\mu \end{aligned} \tag{1.12}$$

$$\begin{aligned} \partial'_\mu \phi' = & \ \frac{\partial \phi'}{\partial X'^\mu} \\ = & \ \frac{\partial \phi}{\partial X'^\mu} \\ = & \ \frac{\partial \phi}{\partial X^\nu} \frac{\partial X^\nu}{\partial X'^\mu} \\ = & \ \frac{\partial X^\nu}{\partial X'^\mu} \partial_\nu \phi \end{aligned} \tag{1.13}$$

In the equations above, it should be kept in mind that, a scalar remains invariant under coordinate transformation, but the functional form of the scalar obviously changes. $\phi = \phi(X^\mu) = \phi'(X'^\mu)$.
We can easily check that the quantity $\dfrac{\partial X^\nu}{\partial X'^\mu}$ corresponds to $((L^T)^{-1})_{\mu\nu}$.

$$\begin{aligned} & \underline{X}' && = L\underline{X} \\ \Rightarrow \ & L^{-1}\underline{X}' && = \underline{X} \\ \Rightarrow \ & \underline{X} && = L^{-1}\underline{X}' \\ \Rightarrow \ & X_\nu && = \sum_\rho (L^{-1})_{\nu\rho} X'_\rho \\ \Rightarrow \ & X_\nu && = \sum_\rho (L^{-1})^T_{\rho\nu} X'_\rho \\ \Rightarrow \ & X_\nu && = \Lambda_\rho{}^\nu X'_\rho \end{aligned} \tag{1.14}$$

[2]Under a coordinate transformation, it is the vector components which change. Hence if we write a vector $A = A^\mu \hat{e}_\mu$, under a transformation, the basis vectors $\hat{e}_\mu$ must transform opposite to the vector components A^μ in order to keep the physical quantity A invariant. Covectors transform oppositely to that of vectors, just like the basis vectors

where

$$\Lambda_\rho{}^\nu = (L^{-1})^T_{\rho\nu} \tag{1.15}$$

As before we can write $\dfrac{\partial X^\nu}{\partial X'^\mu}$ as,

$$\begin{aligned}
\frac{\partial X^\nu}{\partial X'^\mu} &= \partial'_\mu(\Lambda_\rho{}^\nu X'^\rho) \\
&= \Lambda_\rho{}^\nu \partial_\mu X'^\rho \\
&= \Lambda_\rho{}^\nu \delta^\rho{}_\mu \\
&= \Lambda_\mu{}^\nu \qquad (1.16)
\end{aligned}$$

One has to carefully note the position of the indices of Λ.

1.2.2. Tensors of higher rank. We can also have quantities like $A^\mu B^\nu$. Their transformation rule will be given by,

$$\begin{aligned}
A'^\mu B'^\mu &= \Lambda^\mu{}_\rho A^\rho \Lambda^\nu{}_\sigma B^\sigma \\
&= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma A^\rho B^\sigma
\end{aligned}$$

These quantities are second rank tensors. We can have three different kinds of second rank tensors, contravariant, covariant and mixed. They will transform as follows,

$$\begin{aligned}
T'^{\mu\nu} &= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma} \\
T'_{\mu\nu} &= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma T^{\rho\sigma} \\
T'^\mu{}_\nu &= \Lambda^\mu{}_\rho \Lambda_\nu{}^\sigma T^\rho{}_\sigma
\end{aligned}$$

CHAPTER 2

Discovering Electromagnetism

2.1. Lecture 3: *Obtaining* Lorentz Force Law

In this section we shall see, how electromagnetism (well, not the whole of electromagnetism, but atleast the Lorentz force law) follows almost naturally from special relativity.

2.1.1. Least Action Principle.

To start with we shall assume that the only object we have at hand is a point particle. We shall rely on the Principle of Least Action to investigate the motion of the particle. Suppose the particle follows a certain action minimising (extremising to be precise) path in one particular frame. Under a Lorentz transformation, the physical path followed by the particle shouldn't change. Since the action extremising path is a scalar, the simplest[1] choice would be to consider the action a scalar as well.
Given only a point particle and nothing else, the simplest scalar quantity that we can form is $ds^2 = dx^\mu dx_\mu$. Hence the Action of the particle should be of the form.

$$S = -mc\int_a^b ds \tag{2.1}$$

The action integral can be represented as an integral of the Lagrangian with respect to time.

$$S = \int_{t_1}^{t_2} L dt \tag{2.2}$$

The constant mc has been inserted in order to make the action dimensionally equal to angular momentum. Writing the space-time interval ds, as

$$\begin{aligned} ds = & \quad (c^2dt^2 - dx^2 - dy^2 - dz^2)^{\frac{1}{2}} \\ = & \quad c\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \end{aligned}$$

the action integral takes the form

$$S = \int_{t_1}^{t_2} - mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} dt \tag{2.3}$$

Comparing with

$$S = \int_{t_1}^{t_2} L dt$$

[1] Occam's Razor

we can write,

$$L = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \tag{2.4}$$

The quantity

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \tag{2.5}$$

gives the momentum of the particle. The equations of motion for the particle can be obtained from the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) &= \frac{\partial L}{\partial \vec{x}} \\ \Rightarrow \quad \frac{d\vec{p}}{dt} &= 0 \end{aligned} \tag{2.6}$$

and the total energy of the particle is given by the quantity

$$\begin{aligned} H = & \quad \vec{p}\cdot\vec{v} - L \\ = & \quad \frac{mv^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} + mc^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} \\ = & \quad \frac{mc^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \end{aligned} \tag{2.7}$$

2.1.2. Four Potential. That's all we can do using just a point particle and Lorentz invariance. To proceed further towards obtaining Lorentz force law, we now need to bring in another four vector A_μ. We will not treat A_μ as a dynamical variable, instead we will consider it to be fixed from outside without any time-evolution.

Once we have A_μ, let's see what changes we can make to the action integral. The term should be a scalar, involving both A_μ and X_μ. Moreover we also need to have a differential, since we would be doing an integration. Hence the most obvious choice[2] for the additional term would be

$$-q\int_a^b A_\mu dx^\mu$$

Here a scalar q is a parameter which determines the interaction of the particle with the field. Thus the new action integral for the particle is

$$S = \int_a^b -mcds - qA_\mu dx^\mu \tag{2.8}$$

[2]The form of the additional term cannot be fully justified from general considerations alone, it is to some extent a consequence of experimental data.

Separating the time and space part of A,

$$\begin{aligned} A &= \left(\frac{\phi}{c}, \vec{A}\right) \\ A_\mu dx^\mu &= \frac{\phi}{c} cdt - \vec{A}\cdot d\vec{r} \\ &= \left(\phi - \vec{A}\cdot\vec{v}\right) dt \end{aligned} \tag{2.9}$$

Hence,

$$S = \int_{t_1}^{t^2} \left(-mc^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} + q\vec{A}\cdot\vec{v} - q\phi\right) dt \tag{2.10}$$

The new Lagrangian in this case is

$$-mc^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} + q\vec{A}\cdot\vec{v} - q\phi \tag{2.11}$$

2.1.3. Euler-Lagrange equation leading to the Lorentz force law. Let us write down the Euler Lagrange equation for the system using the new Lagrangian.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{\partial L}{\partial \vec{r}} \tag{2.12}$$

The generalised momentum in this case is,

$$\begin{aligned} \vec{\pi} &= \frac{\partial L}{\partial \vec{v}} \\ &= \frac{m\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} + q\vec{A} \\ &= \left(\vec{p} + q\vec{A}\right) \end{aligned} \tag{2.13}$$

Hence,

$$\frac{d\vec{p}}{dt} + q\frac{d\vec{A}}{dt} = -q\vec{\nabla}\phi + q\vec{\nabla}\left(\vec{A}\cdot\vec{v}\right)$$

Writing the individual components[3],

$$\begin{aligned} \frac{dp_i}{dt} &= -q\partial_i\phi + q\partial_i\left(A_j\right)v_j - q(\partial_j A_i)v_j - q\frac{\partial A_i}{\partial t} \\ &= q\left(-\partial_i\phi - \frac{\partial A_i}{\partial t}\right) + q\left(\partial_i A_j - \partial_j A_i\right)v_j \end{aligned} \tag{2.14}$$

[3] A_i refers to the components of the three vector $\vec{A}$

The term $(\partial_i A_j - \partial_j A_i)\, v_j$ can be written as,

$$\begin{aligned}
(\partial_i A_j - \partial_j A_i)\, v_j = &\quad (\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\, v_j \partial_k A_l \\
= &\quad \epsilon_{mij}\epsilon_{mkl} v_j \partial_k A_l \\
= &\quad \epsilon_{ijm} v_j \left(\epsilon_{mkl}\partial_k A_l\right) \\
= &\quad \left[\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right]_i
\end{aligned}$$

Therefore,

$$\frac{dp_i}{dt} = \quad q\left(-\partial_i \phi - \frac{\partial A_i}{\partial t}\right) + q\left[\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right]_i$$

Writing,

$$\begin{aligned}
\left(-\partial_i \phi - \frac{\partial A_i}{\partial t}\right) = &\quad B_i \\
\left(\vec{\nabla} \times \vec{A}\right) = &\quad \vec{E}
\end{aligned}$$

we get,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{2.15}$$

Equation 2.15 is nothing but the Lorentz force law, though it is not in a manifestly Lorentz covariant form.

2.2. Lecture 4: Manifestly Covariant form of the Lorentz Force Law

In the last section we saw how the Lorentz force law arises from Special Relativity and the Least Action Principle. However the final form was not manifestly Lorentz covariant. In this section we shall redo the derivation in a slightly different manner in order to make it so.
Let's start with the same Lagrangian as before.

$$S = \int_a^b -mcds - qA_\mu dx^\mu \tag{2.16}$$

At the extremum, the first order variation in S would vanish.

$$\delta S = 0$$

$$\delta s = \int_a^b [-mc\delta(ds) - q\delta A_\mu dx^\mu - qA_\mu \delta(dx^\mu)] \tag{2.17}$$

The term $\delta(ds)$ can be written as,

$$\delta(ds) = \frac{u_\mu}{c} d(\delta x^\mu)$$

where

$$u_\mu \equiv \frac{dx_\mu}{d\tau}, \quad \tau \text{ being the proper time.}$$

Hence,

$$\begin{aligned} \delta S &= \int_a^b -d[(mu_\mu + qA_\mu)]\delta x^\mu] \\ &\quad + \int_a^b \frac{d}{d\lambda}(mu_\mu + qA_\mu)\delta x^\mu d\lambda - \int_a^b q\partial_\nu A_\mu \frac{dx^\nu}{d\lambda}\delta x^\mu d\lambda \\ &= \int_a^b \left[\frac{d}{d\lambda}(mu_\mu + qA_\mu) - q\partial_\nu A_\mu \frac{dx^\nu}{d\lambda}\right] \quad \delta x^\mu d\lambda = 0 \end{aligned}$$

$$\Rightarrow \quad \frac{dp_\mu}{d\lambda} + q\frac{dA_\mu}{d\lambda} = q\partial_\mu A_\nu \frac{dx^\nu}{d\lambda} \tag{2.18}$$

Since proper time is monotonically increasing, $\frac{d\lambda}{d\tau}$ doesn't diverge and we can multiply by $\frac{d\lambda}{d\tau}$ throughout.

$$\begin{aligned} \frac{dp_\mu}{d\tau} + q\frac{dA_\mu}{d\tau} &= a\partial_\mu A_\nu u^\nu \\ \frac{dp_\mu}{d\tau} + q\partial_\nu A_\mu u^\nu &= a\partial_\mu A_\nu u^\nu \\ \frac{dp_\mu}{d\tau} &= q(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu \end{aligned} \tag{2.19}$$

We call the part in the brackets $(\lambda_\mu A_\nu - \lambda_\nu A_\mu)$ as $F_{\mu\nu}$, an antisymmetric tensor which leads to the manifestly covariant form.

$$
\begin{aligned}
F'_{\mu\nu} &= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma} \\
F_{0i} &= \partial_0 A_i - \partial_i A_0 \\
&= \frac{1}{c}\frac{\partial}{\partial t}(-a_i) - \frac{\partial}{\partial x^i}\frac{\phi}{c} \\
&= \frac{1}{c}\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial a_i}{\partial t}\right) \\
&= \frac{E_i}{c} \\
F_{ij} &= \partial_i A_j - \partial_j A_i \\
&= -(\partial_i a_j - \partial_j a_i) \\
F_{12} &= -B_3 \\
F_{13} &= B_2 \\
F_{23} &= -B_1
\end{aligned}
$$

Thus

$$
F_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{2.20}
$$

and

$$
F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma} \tag{2.21}
$$

Also, if we contract $F_{\mu\nu}$ with the corresponding dual we get a scalar.

$$
F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2/c^2) \tag{2.22}
$$

Since $F_{\mu\nu}$ is an antisymmetric tensor, F_{0i} (the electric field vector) and $\epsilon_{ijk}A_{jk}$ (the magnetic field) transform as vectors.

2.3. Lecture 5: Least Action Principle

2.3.1. A note on the Least Action Principle: In any problem involving a Lagrangian, an action (S) is first written. The least action principle states that for the actual path, this action is stationary. This means that the 1^{st} order variation in S is zero i.e., $\delta S = 0$. In the last lecture we encountered the following equations while deriving the equation of motion for the action S

$$S = \int [-mcds - qA_\mu dx^\mu] \tag{2.23}$$

$$\delta S = \int_{\lambda=0}^{\lambda=1} \delta x^\mu d\lambda [\frac{d}{d\lambda}(mu_\mu + qA_\mu) - q\partial_\mu A_\nu \frac{dx_\nu}{d\lambda}] = 0 \tag{2.24}$$

Note that the limit of integration i.e., the limits of the parameter λ can always be chosen to be from 0 to 1. If we were to choose the parameter to be τ (the proper time), we would run into the hassle of putting different limits for different paths. So we'll stick to a general parameter λ and assume that such a parameter always exists. In order to conclude that the integrand vanishes let us first assume that the integrand is non-zero at atleast one point. Since the integrand is continuous, it must be non-zero around a neighbourhood of that point also (call this region R). We are at freedom to choose δx^μ. We can choose it to be such, that its support is a subset of R and that it is positive in its support (support of a function f(x) is the set of values of x such that f(x)$\neq$0). But then this would make $\delta S > 0$. Thus we must have integrand=0 $\forall x^\mu$ (see figure 1).

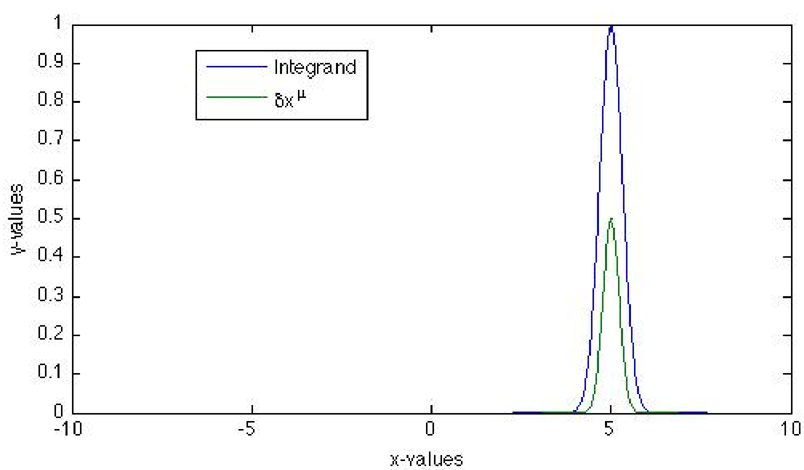


FIGURE 1. Argument showing integrand appearing in the equation $\delta S = 0$ vanishes $\forall x^\mu$

2.3.2. Field Strength Tensor. The manifestly Lorentz covariant form of the solution to equation 1.2 is

$$\frac{dp_\mu}{d\tau} = qF_{\mu\nu}u^\nu \tag{2.25}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Clearly this is an anti-symmetric covariant tensor of rank 2 and transforms as in 2.4.

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{bmatrix} \tag{2.26}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{bmatrix} \tag{2.27}$$

$$F^{'}_{\mu\nu} = {\Lambda_\mu}^\rho {\Lambda_\nu}^\sigma F_{\rho\sigma} = [(L^T)^{-1} F L^{-1}]_{\mu\nu} \tag{2.28}$$

$$F^{\mu\nu} F_{\mu\nu} = 2(B^2 - E^2/c^2) \tag{2.29}$$

2.3.3. A note on the Levi-Civita Tensor. The Levi-Civita tensor in arbitrary dimensions in constructed with the following two basic properties viz.,

(1) It is completely anti-symmetric
(2) $\epsilon^{0123\dots} = 1$

It appears to be a numerical tensor and hence it should not change under any transformation of the co-ordinates. Lets look at how a rank-4 Levi-Civita transforms under Lorentz transformation

$${\Lambda^\kappa}_\pi {\Lambda^\lambda}_\theta {\Lambda^\mu}_\rho {\Lambda^\nu}_\sigma \epsilon^{\pi\theta\rho\sigma} = det(\Lambda) \epsilon^{\kappa\lambda\mu\nu} \tag{2.30}$$

There is an extra factor of $det(\Lambda)$. In three dimensions this problem doesn't appear and $\epsilon^{'ijk} = \epsilon^{ijk}$. We conclude that $\epsilon^{\kappa\lambda\mu\nu}$ must transform differently to maintain its numerical tensor identity. We call it a tensor density. Also this is not a problem if we are concerned with only proper orthochronous Lorentz transformations. 3.2 shows how $\epsilon^{\kappa\lambda\mu\nu}$ actually transforms.

$$\epsilon^{'\kappa\lambda\mu\nu} = \frac{1}{\det(\Lambda)} {\Lambda^\kappa}_\pi {\Lambda^\lambda}_\theta {\Lambda^\mu}_\rho {\Lambda^\nu}_\sigma \epsilon^{\pi\theta\rho\sigma} = \epsilon^{\kappa\lambda\mu\nu} \tag{2.31}$$

Now consider the tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$. This is a covariant tensor density of rank two since it is the result of the tensor product of a rank-4 contravariant tensor density and a rank-2 covariant tensor.

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B_1 & -B_2 & B_3 \\ -B_1 & 0 & E_3/c & -E_2/c \\ B_2 & -E_3/c & 0 & -E_1/c \\ -B_3 & E_2/c & E_1/c & 0 \end{bmatrix} \tag{2.32}$$

2.4. List of Exercises

1.

$$L = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How does $F^{\mu\nu}$ transform under this? Use this to find out how $\vec{E}$ and $\vec{B}$ transform?

2.5. Lecture 6: Electric field of an Uniformly Moving Point Charge

2.5.1. Some digression - Electric Field of an Uniformly Moving Point Charge. Let S be a frame in which $B = 0$. S$^{'}$ be another frame which is moving relative to S along the negative x-axis with a uniform speed βc. We look at an ideal parallel plate capacitor whose plates are parallel to the xy-plane and is stationary in the frame S. In S, there is only a z-component of electric field ($E_z = \sigma/\epsilon_0$). In S$^{'}$, this $E_z \to E_z^{'} = \gamma E_z$. This can be derived from the $F_{\mu\nu}$ tensor and can also be understood intuitively. In S$^{'}$, the lengths along the x-axis appear contracted. So the charge density goes up by a factor of γ in this frame. Thus the field E_z goes up by the factor γ. The component of $\vec{E}$ parallel to the motion of S$^{'}$ (relative to S) remains unchanged.

$$E_{\parallel}^{'} = E_{\parallel} \tag{2.33}$$

$$E_{\perp}^{'} = \gamma E_{\perp} \tag{2.34}$$

Now consider a point charge Q placed at the origin of S. In S, the field is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \tag{2.35}$$

In S$^{'}$ (moving with a speed βc along the negative x-axis), the co-ordinates are

$$x = \gamma(x^{'} - \beta ct^{'}) \tag{2.36}$$

$$y = y^{'} \tag{2.37}$$

$$z = z^{'} \tag{2.38}$$

and the field looks like

$$E_x^{'} = E_x \tag{2.39}$$

$$E_y^{'} = \gamma E_y \tag{2.40}$$

$$E_z^{'} = \gamma E_z \tag{2.41}$$

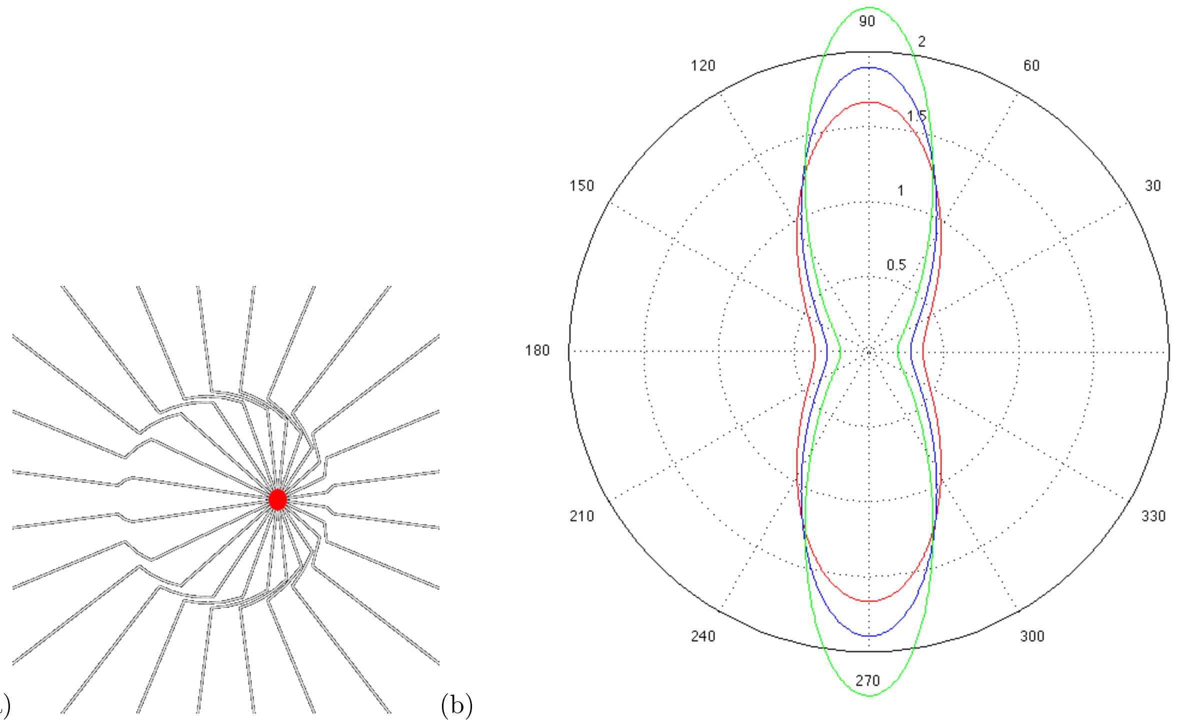
$$\vec{E}^{'} = \frac{\gamma Q}{4\pi\epsilon_0}\frac{(\vec{r^{'}} - \vec{\beta}ct^{'})}{[\gamma^2(x' - \beta ct')^2 + y'^2 + z'^2]^{3/2}} \tag{2.42}$$

At t$^{'}$=0, the magnitude of the field as seen from the frame S$^{'}$ can be written as

$$|\vec{E}^{'}|_{t'=0} = \frac{Q}{4\pi\epsilon_0 r'^2}\frac{(1-\beta^2)}{[1-\beta^2 sin^2\theta']^{3/2}} \tag{2.43}$$

where $\theta^{'}$ is the angle between $\vec{\beta}$ and $(\vec{r^{'}} - \vec{\beta}ct^{'})$. Thus there is a distribution of the strength of electric field when seen from the frame S$^{'}$. For a given $r^{'}$, it is maximum at the points on the lines perpendicular to the line of motion of Q in S$^{'}$.

The electric field of a charged particle initially at rest and suddenly accelerating to a speed v within a time Δt is also very interesting. There are three distinct regions in the space around the particle when we look at the behaviour of E at later times. At time t the particle is at the position vt. Region I is a sphere of radius ct around the origin. In this region, the field is that of a moving charged particle. The region II is of width $c\Delta t$. It consists of tangential electric field lines



(a) (b)

FIGURE 2. (a) The field of a charge initially at rest and suddenly accelerating at t=0 and thereafter moving with a constant speed v. (b) The angular dependence of the field strength of a moving charged particle. The distance of the curve from the origin is proportional to the electric field strength at that angle(θ) for a given $r^{'}$. The red, blue and green curves are for $\beta = 0.8$, 0.85 and 0.9 respectively. The curves slowly flatten to the circle as $\beta \rightarrow 0$ and similarly grows in strength near the poles for increasing $\beta \rightarrow 1$.

which join the field lines of region I with region III. Region III has field lines of that of a charge stationed at the origin since the information that the charge has started moving has not arrived there yet.

2.6. Lecture 7: Maxwell's Equations from the Field Tensor

2.6.1. The Maxwell's Equation.

2.6.1.1. *Gauge Invariance.*

$$A^\mu = \left(\frac{\phi}{c}, \vec{a}\right) \tag{2.44}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.45}$$

Under a gauge transformation,

$$A_\mu \rightarrow A'_\mu + \partial_\mu \chi \tag{2.46}$$

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu} \tag{2.47}$$

provided,

$$\partial_\mu \partial_\nu \chi = \partial_\nu \partial_\mu \chi \tag{2.48}$$

2.6.2. Obtaining the Maxwell's Equations. The action integral as obtained earlier is,

$$S = \int \left(-mcds - qA_\mu dx^\mu - \alpha F_{\mu\nu} F^{\mu\nu} d^4x\right) \tag{2.49}$$

For a localised charge δq, $\delta q dx^\mu$,

$$\begin{aligned} \delta q dx^\mu = & \quad \rho dv dx^\mu \\ = & \quad \rho dv dt \tfrac{dx^\mu}{dt} \\ = & \quad \rho d^4x \tfrac{dx^\mu}{dt} \\ = & \quad \rho \tfrac{dx^\mu}{dt} d^4x \end{aligned}$$

The term $\rho \frac{dx^\mu}{dt}$ is a four vector and can be represented by j^μ.

$$\begin{aligned} j^\mu = & \quad \rho \tfrac{dx^\mu}{dt} \\ = & \quad (\rho c, \rho \vec{v}) \end{aligned}$$

So in the action integral, the electromagnetic part can be re-written as,

$$S_{em} = -\int j^\mu A_\mu d^4x - \alpha \int F_{\mu\nu} F^{\mu\nu} d^4x \tag{2.50}$$

The variation in the electromagnetic part becomes,

$$\delta S = -\int j \delta A_\mu d^4x - \alpha \int \delta(F_{\mu\nu} F^{\mu\nu}) d^4x \tag{2.51}$$

The variation in $F_{\mu\nu} F^{\mu\nu}$,

$$\delta(F_{\mu\nu} F^{\mu\nu}) = 2(\delta F_{\mu\nu}) F^{\mu\nu} \tag{2.52}$$

$$\begin{aligned} & (\delta F_{\mu\nu}) F^{\mu\nu} + F_{\mu\nu} \delta(F^{\mu\nu}) \\ = & \quad (\delta F_{\mu\nu}) F^{\mu\nu} + F^{\mu\nu} \delta(F_{\mu\nu}) \\ = & \quad 2(\delta F_{\mu\nu}) F^{\mu\nu} \end{aligned}$$

Now,

$$\begin{aligned}\delta F_{\mu\nu} = & \quad \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ = & \quad \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)\end{aligned}$$

Hence,

$$\begin{aligned}& \partial_\mu(\delta A_\nu)F^{\mu\nu} - \partial_\nu(\delta A_\mu)F^{\mu\nu} \\ = & \quad \partial_\mu(\delta A_\nu)F^{\mu\nu} + \partial_\nu(\delta A_\mu)F^{\nu\mu} \\ = & \quad 2\partial_\mu(\delta A_\nu)F^{\mu\nu}\end{aligned}$$

In deriving the above expression, we first utilised the antisymmetry of $F^{\mu\nu}$ and then simply flipped the dummy indices μ & ν in the second term.
So finally we have,

$$\begin{aligned}2(\delta F_{\mu\nu}F^{\mu\nu}) = & \quad 2(2\partial_\mu(\delta A_\nu)F^{\mu\nu}) \\ = & \quad 4\partial_\mu(\delta A_\nu)F^{\mu\nu} \\ = & \quad 4\partial_m u(\delta A_\nu F^{\mu\nu}) \\ & -4\delta A_\nu \partial_\mu F^{\mu\nu}\end{aligned}$$

Making the variation vanish,

$$-\int j^\mu \delta A_\mu d^4x - 4\alpha \int \partial_\mu(\delta A_\nu F^{\mu\nu})d^4x + 4\alpha \int \partial_\nu f^{\mu\nu}\delta A_\mu d^4x = 0$$

The second integral vanishes, as it is a volume integral of a 4-divergence. Hence,

$$\int \left(j^\mu + 4\alpha\partial_\nu F^{\mu\nu}\right)\delta A_\mu d^4x = 0 \tag{2.53}$$

Since the integral must vanish for all arbitrary variations, the quantity $(j^\mu + 4\alpha\partial_\nu F^{\mu\nu})$ must vanish identically.

$$\partial_\nu F^{\mu\nu} = -\frac{1}{4\alpha}j^\mu \tag{2.54}$$

The above equation contains two of the Maxwell's equation.
Equating the zeroth component of j,

$$\begin{aligned}\partial_\nu F^{0\nu} = -\frac{1}{4\alpha}j^0 = -\frac{c}{4\alpha}\rho \\ F^{0i} = -\frac{E^i}{c}\end{aligned}$$

Hence,

$$\begin{aligned}-\tfrac{\vec{\nabla}\cdot\vec{E}}{c} = & \quad -c\frac{c}{4\alpha}\rho \\ \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = & \quad \frac{c^2}{4\alpha}\rho = \frac{\rho}{\epsilon_0}\end{aligned}$$

which is the Gauss Law.
Equating the other three components of j,

$$\partial_\nu F^{i\nu} = -\frac{1}{4\alpha}j^i \tag{2.55}$$

leads to the Ampere's law,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.56}$$

The other two Maxwell's equation,

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.57}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial B}{\partial t} = 0 \tag{2.58}$$

follows from the definition of $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.59}$$