Lienard-Wiechart Potentials

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I. INTROUCTION

The Lienard-Wiechart potentials have already been derived. In the relativistically covariant form they were written as:

$$A^{\mu}(x) = \frac{\mu_0}{4\pi} qc \left[\frac{\tilde{U}^{\mu}(\tau)}{\tilde{U}^{\nu}(\tau)[r - \tilde{w}(\tau)]_{\nu}} \right]_{\tau=\tau_0} \tag{1}$$

We now write these potentials in the non-covariant form(as was derived in class).We have already seen that the motion of the charge satisfies the condition:-

$$r^0 - \tilde{w^0}(\tau_0) = |\mathbf{r} - \tilde{\mathbf{w}}(\tau_{\mathbf{0}})| = R \tag{2}$$

This implies

$$\begin{aligned} V.(r - \tilde{w}(\tau_0) &= \tilde{U}_0[r^0 - \tilde{w^0}(\tau_0] - \tilde{\mathbf{U}}.[\mathbf{r} - \tilde{\mathbf{w}}(\tau_{\mathbf{0}})] \\ &= \gamma c R - \gamma \tilde{\mathbf{U}}.\mathbf{n} R \\ &= \gamma c R(1 - \beta.\mathbf{n}) \end{aligned} \tag{3}$$

where $\mathbf{n}$ is the unitvector in the direction $\mathbf{r} - \tilde{\mathbf{w}}(\tau)$ and $\beta = \tilde{\mathbf{U}}(\tau)/c$. Hence, in the relativistically non-covariant form, this can be written as

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{e}{(1 - \beta.\mathbf{n})R} \right]_{\tau_0} \tag{4}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \left[\frac{e\beta}{(1 - \beta.\mathbf{n})R} \right]_{\tau_0} \tag{5}$$

II. DERIVING THE ELECTROMAGNETIC FIELDS

The electromagnetic fields $F^{\alpha\beta}$(x) can be calculated directly from eq.(1).However,in this case the calculations become far simpler if we go back to the integral form of eq.(1)

$$A^{\mu} = \frac{\mu_0}{2\pi} qc \int d\tau \tilde{U}^{\mu} \theta(r^0 - \tilde{w^0}(\tau)) \delta\left((r - \tilde{w}(\tau))^2\right) \tag{6}$$

In order to determine the fields, we carry out a partial derivative with respect to the observation point x.Now,such a differentiation when acting on the theta function would produce $\delta[r^0 - \tilde{w^0}(\tau]$ and so constrain the delta function to be $\delta(-R^2)$.There will be no contribution from this differentiation except ar $R = 0$. Excluding that point from consideration we get

$$\partial^{\nu} A^{\mu} = \frac{\mu_0}{2\pi} qc \int d\tau \tilde{U}^{\mu} \theta(r^0 - \tilde{w^0}(\tau)) \partial^{\nu} \delta\left((r - \tilde{w}(\tau))^2\right) \tag{7}$$

In order to take the derivative we perform the following trick:

$$\partial^{\nu}\delta[f] = \partial^{\nu} f.\frac{d}{df}\delta[f] = \partial^{\nu} f.\frac{d\tau}{df}\frac{d}{d\tau}\delta[f] \tag{8}$$

where f is $(r - \tilde{w}(\tau))^2$.The differentiation would yield:

$$\partial^{\nu}\delta[f] = -\frac{(r - \tilde{w})^{\nu}}{\tilde{U}.(r - \tilde{w})} \tag{9}$$

This result is inserted into eq.(7).After that an integration is performed taking the delta function as the first function.The result can be written down as follows:

$$\partial^{\nu} A^{\mu} = \frac{\mu_0}{2\pi} qc \int d\tau \frac{\partial}{\partial \tau}[\frac{(r-\tilde{w})^{\nu}\tilde{U}^{\mu}}{\tilde{U}.(r-\tilde{w})}]\theta(r^0 - \tilde{w^0}(\tau))\delta\left((r-\tilde{w}(\tau))^2\right) \tag{10}$$

In this integration, the derivative of the theta function doesn't contribute.The form of this equation is the same as that of eq.(6) with $\tilde{U}^{\mu}$ being replaced by the derivative term. Now,the result of eq.(6) is written in eq.(1).Hence,we can directly read off the result. The field strength tensor is

$$F^{\nu\mu} = \frac{\mu_0}{4\pi}\left[\frac{qc}{\tilde{U}.(r-\tilde{w})}\frac{\partial}{\partial\tau}[\frac{(r-\tilde{w})^{\nu}\tilde{U}^{\mu} - (r-\tilde{w})^{\mu}\tilde{U}^{\nu}}{\tilde{U}.(r-\tilde{w})}]\right] \tag{11}$$

The whole expression is evaluated at the retarded proper time τ_0. In order to explicitly determine the electric and magnetic fields as functions of the velocity and accelration,we need to use the following result:-

$$\frac{d\tilde{U}}{d\tau} = [c\gamma^4\beta.\dot{\beta}, c\gamma^2\dot{\beta} + c\gamma^4\beta(\beta.\dot{\beta})] \tag{12}$$

Using this expression,we can write the electric and magnetic fields in their more familiar form as had been derived in class:-

$$\mathbf{E}(\mathbf{r},\mathbf{t}) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\mathbf{R}.\mathbf{u})^{\mathbf{3}}}[(c^2 - u^2\mathbf{u} - \mathbf{R}\times(\mathbf{u}\times\mathbf{a})] \tag{13}$$

$$\mathbf{B}(\mathbf{r},\mathbf{t} = \frac{1}{c}\mathbf{n}\times\mathbf{E} \tag{14}$$

In the above

$$u = c\mathbf{n} - \mathbf{U} \tag{15}$$

$$R = r - \tilde{w} \tag{16}$$

III. CONCLUSION

The Lienard-Wiechert potentials play a pivotal role in the analysis of power radiated by a moving charge.This method avoids the lengthy and cumbersome approach of Vector Calculus and shows a very elegant way of arriving at the Lienard-Wiechert potentials and also the fields arising from these potentials.

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[1] Classical Electrodynamics-J.D.Jackson