PH3202 Problem Set 1

Q 1) Given $\vec{A} = (3x^2 + 6y)\,\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, calculate the line integral $\int_\Gamma \vec{A} \cdot d\vec{r}$ for the following paths from $(0, 0, 0)$ to $(1, 1, 1)$:

(a) $x = t$, $y = t^2$, $z = t^3$

(b) straight line joining these two points

(c) path formed by straight line segments joining $(0, 0, 0)$ to $(0, 0, 1)$to $(0, 1, 1)$to $(1, 1, 1)$.

Q 2) Evaluate $\int_\Sigma \vec{F} \cdot \hat{n}\, dS$ where

(a) $\vec{F} = y\hat{\imath} + 2x\hat{\jmath} - z\hat{k}$ and Σ is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

(b) $\vec{F} = 4xz\hat{\imath} + xyz^2\hat{\jmath} + 3z\hat{k}$ and Σ is the surface of the entire region above the XY plane bounded by the cone $z^2 = x^2 + y^2$and the plane $z = 4$.

Q 3) Under a particular rotation of coordinate systems, we have the coordinates changing by

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\sin\theta + y\cos\theta, \quad z' = z$$

Consider two vectors $\vec{A}$ and $\vec{B}$, whose components transform the same way.

a) Show that $\vec{A} \cdot \vec{B}$ does not transform under the rotation - *i.e.* it is a scalar.

b) Show that the three components of $\vec{A} \times \vec{B}$ transform like the coordinates.

c) Show that $\nabla \cdot \vec{A}$ does not change under this transformation.

d) Show that $\nabla \times \vec{A}$ components transform like those of a vector.

Q 4) Consider a finite straight line charge with a uniform charge density. Describe all the information that you can obtain about the nature of its electric field from symmetry considerations. Note that we are talking of the electric field as a whole - at all points!

Q 5) What can you say about the nature of the magnetic field caused by a circular ring of current from symmetry considerations alone?

For the necessary background for questions 4 and 5, you can take a look at my blog article It is obvious fom symmetry that

Q 6) Consider the function

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

(a) Show by an explicit calculation that

$$\nabla^2 f = 0 \qquad \text{for } (x, y, z) \neq (0, 0, 0)$$

Both the function, and its Laplacian diverge at the origin

(b) Again by explicit calculation show that the surface integral

$$\oint_\Sigma \nabla f \cdot d\vec{S} = -4\pi$$

where Σ is the unit sphere $x^2 + y^2 + z^2 = 1$.

(c) Hence argue that

$$\nabla^2 f = -4\pi\delta^3(0)$$

(d) The method above is hardly very mathematically rigorous. To do better, consider the function

$$f_\epsilon(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 + \epsilon^2}}$$

Calculate $\nabla^2 f_\epsilon$, which is well defined on all of $\mathbb{R}^3$.

(e) Show that $\int \nabla^2 f_\epsilon d^3r$ where the volume integration over all space is independent of ϵ.

(f) Show that for $(x, y, z) \neq (0, 0, 0)$

$$\lim_{\epsilon \to 0} \nabla^2 f_\epsilon = 0$$

and hence argue that

$$\lim_{\epsilon \to 0} \nabla^2 f_\epsilon = -4\pi\delta^3(0)$$