PH3202 Problem Set 4

Q 1)a) Show that the function $E(x) = (x^2 - 1)^l$ obeys

$$\left(x^2 - 1\right) \frac{dE}{dx} = 2lxE(x)$$

b) Differentiate both sides of the equation above $l + 1$ times with respect to x show that the function

$$F(x) = \frac{d^l}{dx^l} \left(x^2 - 1\right)^l$$

satisfies the Legendre equation.

c) Show that $F(0) = 2^l l!$ and hence justify Rodrigues formula for Legendre polynomials.

Q 2) a) Differentiate the Legendre equation $|m|$ times (where $m = 0, \pm 1, \ldots, \pm l$) to show that the function

$$G(x) = \frac{d^{|m|}}{dx^{|m|}} P_l(x)$$

obeys the second order differential equation

$$\left(x^2 - 1\right) \frac{d^2G}{dx^2} + 2x\left(|m| + 1\right) \frac{dG}{dx} + \left(|m|\left(|m| + 1\right) - l\left(l + 1\right)\right) G = 0$$

b) Show that this is the same differential equation as that obeyed by $(x^2 - 1)^{|m|/2} F(x)$ where $F(x)$ satisfies the associated Legendre equation.

Q 3) a) Starting from the generating function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

of Legendre polynomials, show by differentiation with respect to t of both sides that

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n\left(x\right)t^{n-1}$$

b) Hence prove Bonnet's recursion formula

$$\left(n+1\right)P_{n+1}\left(x\right) = \left(2n+1\right)xP_n\left(x\right) - nP_{n-1}\left(x\right)$$

c) Use this, along with $P_0\left(x\right)=1$ and $P_1\left(x\right)=x$ to determine the Legendre polynomials from $l=2$ to $l=6$.

d) Use the recursion formula to show that

$$P_{l+2}\left(0\right) = -\frac{l+1}{l+2}P_l\left(0\right)$$

and hence derive an explicit expression for $P_l\left(0\right)$. Show that this is the same as the expression you can obtain from the Rodrigues formula.

Q 4 a) Using Rodrigues formula, show that

$$\int_0^1 P_n\left(z\right)dz = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for other even } n \\ \left(-1\right)^{(n-1)/2}\dfrac{(n-1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} & \text{for odd } n \end{cases}$$

b) Show that for $n>0$ this can be written as

$$\int_0^1 P_n\left(z\right)dz = \frac{P_{n-1}\left(0\right)}{n+1}$$

c) Use Bonnet's recursion formula to derive

$$\int_{-1}^1 |x|\,P_n\left(x\right)dx = -\frac{2P_n\left(0\right)}{\left(n-1\right)\left(n+2\right)}$$

Q 5) a) Show that if you assume that the delta function can be written in the form of a Fourier-Legendre series, then it takes the form

$$\delta(z) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(0) P_n(z)$$

b) In the class we discussed the apparent paradox that arises when we try to determine the continuity of the radial component of the electric field on the sphere $r = a$, due to a uniformly charged ring of radius a. Show that the paradox can be resolved using the result of part a above.

Q 6) By direct integration find the potential everywhere on the axis of a uniformly charged disc of radius a carrying a surface charge density of σ_0. Use this result and the fact that the potential satisfies Laplace equation everywhere away from the disc to find the potential everywhere.

Comment on the continuity (or otherwise) of this potential and its first derivatives for $r < a$, $\theta = \frac{\pi}{2}$, as well as on the sphere $r = a$.

Q 7) Find the potential everywhere in the upper half plane $z \geq 0$ for the problem described in Question number 3 of Problem sheet 3.

Q 8) Consider the problem of two hemispheres, insulated from each other by a thin ring (but otherwise forming a complete sphere) and maintained at potentials of $+V_0$ and $-V_0$, respectively. Find the surface charge density on the sphere and the net charge on it.

Q 9) Find the potential everywhere for a system of two concentric spheres of radii a and b ($a < b$). It is specified that the potential on the inner sphere is given by

$$V(a, \theta) = V_0 \cos^2 \theta$$

and that on the outer sphere by

$$V(b, \theta) = V_0 \cos \theta$$

Find the total charge on each sphere.

Q 10)a) Find the potential everywhere due to a hemispherical shell that carries a uniform surface charge density by

i) first calculating the potential on the axis and using it to find the potential everywhere.

ii) or by starting with the general form

$$V(r,\theta) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) & \text{for } r < a \\ \displaystyle\sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta) & \text{for } r > a \end{cases}$$

and by imposing the continuity of V at $r = a$, along with the discontinuity in the derivative implied by the surface charge density.

b) Use your result to show that the electric field everywhere on the flat cap of the hemisphere (note that there is no charge there!)) is normal to the cap. *Hint : for this you do not need to know is which coefficients are non-zero!*

c) Part (b) above is actually from Purcell's beautiful book on electrodynamics. There is a wonderful argument that allows you to solve it without any maths! Try to find that!