CHAPTER 1

Polarisation

(This report was prepared by Abhishek Dasgupta and Arijit Haldar based on notes in Dr. Ananda Dasgupta's Electromagnetism III class)

Topics covered in this chapter are the Jones calculus, Stokes parameters, the Poincaré sphere and Mueller calculus.

We consider light propagating in the z direction.

$$\begin{aligned} \mathbf{E} &= (E_x\hat{x} + E_y\hat{y})e^{i(kz-\omega t)} \\ &= (|E_x|e^{i\phi_x}\hat{x} + |E_y|e^{i\phi_y}\hat{y})e^{i(kz-\omega t)} \\ &= E_{\text{eff}}(A\hat{x} + Be^{i\delta}\hat{y})e^{i(kz-\omega t+\phi_x)} \end{aligned}$$

where E_{eff} is $\sqrt{|E_x|^2 + |E_y|^2}$ and $A = |E_x|/E_{\text{eff}}$ and $B = |E_y|/E_{\text{eff}}$ and thus $|A|^2 + |B|^2 = 1$. It is important to note that E_{eff} is not the amplitude of the wave. If the polarised light was replaced by a *single* plane polarised light and it had the same energy then E_{eff} would be the amplitude of that wave. We then write the components of the electric field as

$$\begin{aligned} E_x &= AE_{\text{eff}}\cos(kz-\omega t+\phi_x) \\ E_y &= BE_{\text{eff}}\cos(kz-\omega t+\phi_x+\delta) \end{aligned}$$

The Jones vector representation of this state would be

$$V = \begin{pmatrix} A \\ Be^{i\delta} \end{pmatrix}$$

Jones vectors are normalised ($V^*V = I$). By convention the top entry is real. The overall phase factor is immaterial and only the phase difference matters. We shall now obtain a relation between E_x and E_y by eliminating θ from the two equations obtained earlier.

$$\begin{aligned} \frac{E_x}{AE_{\text{eff}}} &= \cos\theta \\ \frac{E_y}{BE_{\text{eff}}} &= \cos\theta\cos\delta - \sin\theta\sin\delta \end{aligned}$$

Eliminating θ,

$$\begin{aligned}\left(\frac{E_x}{AE_{\text{eff}}}\right)^2+\left(\frac{E_x}{AE_{\text{eff}}}\right)^2\cot^2\delta+\left(\frac{E_y}{BE_{\text{eff}}}\right)^2\csc^2\delta-\frac{2E_xE_y}{ABE_{\text{eff}}^2}\csc\delta\cot\delta &= 1\\ \frac{E_x^2}{A^2}+\frac{E_y^2}{B^2}-\frac{2E_xE_y\cos\delta}{AB} &= E_{\text{eff}}^2\sin^2\delta\end{aligned}$$

The left hand side can be expressed as

$$\begin{pmatrix} E_x & E_y \end{pmatrix}\begin{pmatrix} \frac{1}{A^2} & -\frac{\cos\delta}{AB} \\ -\frac{\cos\delta}{AB} & \frac{1}{B^2} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

First, we note that the matrix linking the row and column forms of the electric field vector is symmetric, hence it can be diagonalised. Also the trace and determinant is positive, making E_x and E_y points on an ellipse which is rotated about the origin, the rotation being specified by the above matrix.

1.1. Jones formalism

The simplest case is that of the polariser which selects and allows light propagating in a particular direction to pass. A physical realisation of the polariser is a stretched polymer coated with iodine. The electrons in iodine vibrate along the stretched polymer and emits light thus absorbing light propagating in that direction. The light emitted from the electrons interferes along the line of the polymer cancelling itself out. Only light propagating perpendicular to the stretched polymer passes.

The polymer selects a particular direction; for example if it selects light along the x axis, then it transforms light having electric field of the form $E_x\hat{x}+E_y\hat{y}\rightarrow E_x\hat{x}$. The Jones matrix corresponding to the polariser is one which effects this transformation. We cannot preserve normalisation during the transformation as that would lead to nonlinearity of the Jones matrices. Thus the polariser can be represented by the Jones matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The action of the Jones matrix is then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ Be^{i\delta} \end{pmatrix}=\begin{pmatrix} A \\ 0 \end{pmatrix}$$

Thus the Jones matrix for the polariser is essentially a projection operator or a matrix. Jones matrices can also be singular as in this case.

1.1.1. Jones matrix for a polariser inclined at angle θ. Suppose that the polariser is inclined at angle θ to the x axis. Let the axis along the polariser be represented by the unit vector $\hat{e_1}$ and the axis perpendicular to this represented by $\hat{e_2}$. Only light propagating along $\hat{e_1}$ can pass through the polariser. Then the incoming light ($\cos\theta$ denoted by c and $\sin\theta$ denoted by s respectively)

$$\begin{aligned} E_x\hat{x} + E_y\hat{y} &= E_x(c\hat{e_1} - s\hat{e_2}) + E_y(s\hat{e_1} + c\hat{e_2}) \\ &\to (E_xc + E_ys)\hat{e_1} \\ &= (E_xc + E_ys)(c\hat{x} + s\hat{y}) \\ &= (E_xc^2 + E_ycs)\hat{x} + (E_ys^2 + E_xcs)\hat{y} \end{aligned}$$

Thus the Jones matrix for a polariser inclined at an angle θ is

$$\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

1.1.2. Jones matrix for a general phase shifter. Now we consider the more general case of phase shifters which introduce an extra relative phase ξ between the two orthogonal components of light. ξ is a complex quantity. The previous case can be obtained from this by setting $\xi = 0$. The final state of light after passing through the phase shifter is

$$(E_xc + E_ys)\hat{e_1} + \xi(-E_xs + E_yc)\hat{e_2}$$

Expressing this in the xy basis, this becomes

$$\begin{aligned} &[(E_xc + E_ys)c - \xi(-E_xs + E_yc)s]\hat{x} \quad + \\ &[(E_xc + E_ys)s + \xi(-E_xs + E_ys)c]\hat{y} \end{aligned}$$

the corresponding Jones matrix is

$$\begin{pmatrix} \cos^2\theta + \xi\sin^2\theta & \cos\theta\sin\theta(1-\xi) \\ \cos\theta\sin\theta(1-\xi) & \sin^2\theta + \xi\cos^2\theta \end{pmatrix}$$

Now that we've got the Jones matrix for a general phase shifter we can get the Jones matrices for **half-wave** and **quarter-wave** plates. A half-wave plate puts and additional phase shift of π, thus in this case $\xi = -1$. Essentially the half-wave plate *reflects* one component of light along the axis of the half-wave plate. Thus

$$J_{HW} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

We see that J_{HW} has a determinant of -1 which is consistent with that of a reflection matrix.

A **quarter-wave** plate puts a phase shift of $\pi/2$ ($\xi = i$). Thus

$$J_{QW} = \begin{pmatrix} \cos^2\theta + i\sin^2\theta & \cos\theta\sin\theta(1-i) \\ \cos\theta\sin\theta(1-i) & \sin^2\theta + i\cos^2\theta \end{pmatrix}$$

Applying J_{QW} corresponding to a plate inclined at $\pi/4$ on a plane polarised light we get

$$\begin{aligned} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} A+B+(A-B)i \\ A+B-(A-B)i \end{pmatrix} \\ &= \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned}$$

Thus a quarter wave plate changes a linearly polarised light to a circularly polarised light.

1.2. Stokes parameters

While the Jones formalism is perfect for describing fully polarised light, it can not describe mixed polarisation states. Also the components of the Jones vector are not observable quantities; what is observable are the intensities. Stokes introduced a method of describing polarisation using only observable quantities via the Stokes parameters

$$\begin{aligned} s_0 &= I_{xx} + I_{yy} \\ s_1 &= I_{xx} - I_{yy} \\ s_2 &= I_{xy} + I_{yx} \\ s_3 &= i(I_{xy} - I_{yx}) \end{aligned}$$

which can be written in matrix notation as $S = A\tilde{\tilde{I}}$ where S denotes the Stokes vector, $\tilde{\tilde{I}}$ denotes the intensity vector $\begin{pmatrix} I_{xx} & I_{xy} & I_{yx} & I_{yy} \end{pmatrix}$ and

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix}$$

The intensity I_{ij} is the expectation value of $E_i^* E_j$.

The coherence matrix or the correlation matrix is defined as

$$\tilde{I} = \begin{pmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle \\ \langle E_y^* E_x \rangle & \langle E_x^* E_y \rangle \end{pmatrix} = \left\langle \begin{pmatrix} E_x^* \\ E_y^* \end{pmatrix} \begin{pmatrix} E_x & E_y \end{pmatrix} \right\rangle$$

The right hand side can be written as $\left\langle \tilde{E}^* \tilde{E}^T \right\rangle$. The matrix $\tilde{I}$ is Hermitian. From this matrix, we shall define a measure for polarisation. We

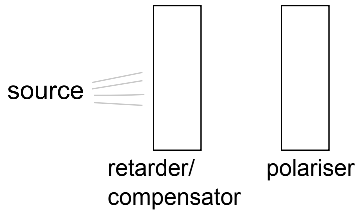

FIGURE 1.2.1. Apparatus for measuring components of the $\tilde{I}$ matrix. The retarder introduces the phase shift and the polariser is inclined at an angle of θ.

note that $\left\langle |c_1 E_x + c_2 E_y|^2 \right\rangle \geq 0$, implying

$$|c_1|^2 I_{xx} + |c_2|^2 I_{yy} + c_1^* c_2 I_{xy} + c_1 c_2^* I_{yx} \geq 0$$

Denoting $\lambda \equiv c_2/c_1$ and dividing throughout by $|c_1|^2$ we get

$$\begin{aligned} I_{xx} + \lambda^2 I_{yy} + \lambda I_{xy} + \lambda^* I_{yx} &\geq 0 \\ I_{xx} + I_{yy} \left| \lambda + \frac{I_{xy}}{I_{yx}} \right|^2 - \frac{|I_{xy}|^2}{I_{yy}} &\geq 0 \end{aligned}$$

The left hand side has minimal positive value when $\lambda + I_{xy}/I_{yx} = 0$, thus

$$|I_{xy}|^2 \leq I_{xx} I_{yy}$$

We can now define a measure for polarisation i_{xy} given by

$$i_{xy} = \frac{|I_{xy}|}{\sqrt{I_{xx} I_{yy}}}$$

The measure of polarisation i_{xy} always lies between 0 and 1, 0 representing perfectly unpolarised light while 1 represents perfectly polarised light.

To find i_{xy} we need to know the intensities so that we can construct the $\tilde{I}$ matrix. We can measure the intensities using a setup as shown in the figure 1.2.1. A general electric field is given as $\mathbf{E}(t) = (E_x(t) e^{i\epsilon_1} \cos\theta + E_y(t) e^{i\epsilon_2} \sin\theta)\hat{n}$. Then the intensity is given by

$$I_{xx} \cos^2\theta + I_{yy} \sin^2\theta + (I_{xy} e^{i\delta} + I_{yx} e^{-i\delta}) \cos\theta \sin\theta$$

We put $\theta = 0, \frac{\pi}{2}$ to get I_{xx} and I_{yy} from which we can obtain the first two Stokes parameters. Using polarisers inclined at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ respectively we can get $s_2 = I_{xy} + I_{yx} = I_{\nearrow} - I_{\nwarrow}$. For the last Stokes parameter, we use quarter wave plates inclined at an angle of $\frac{\pi}{4}$, using one quarter wave plate for right circularly polarised and another for left circularly polarised. Then $I_{\circlearrowright} - I_{\circlearrowleft} = i(I_{xy} - I_{yx})$.

Referring to the Stokes parameters we see that

$$s_0^2 - s_1^2 = 4 I_{xx} I_{yy} \geq 4|I_{xy}|^2 = s_2^2 + s_3^2$$

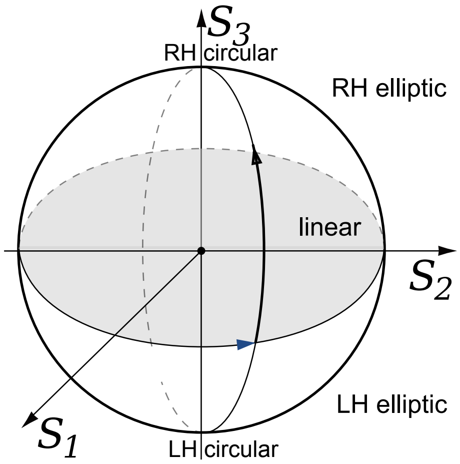

FIGURE 1.2.2. The Poincaré sphere showing the various polarisation states. Points on the surface of the sphere are fully polarised, while the centre represents fully unpolarised light.

Thus $s_0^2 - s_1^2 - s_2^2 - s_3^2 \geq 0$. We can think of (s_1, s_2, s_3) as points residing within the surface of a sphere of radius s_0. This sphere is known as the **Poincaré sphere**. Points on the surface of the sphere represent fully polarised light states. The centre of the sphere is the fully unpolarised state. All points on the equatorial diameter ($s_3 = 0$) represent linearly polarised states while the poles of the sphere $(0, 0, s_0)$ and $(0, 0, -s_0)$ represent right handed and left handed circularly polarised light respectively.

Another way of representing the polarisation of a light is through the determinant of $\tilde{I}$. For a perfectly polarised light, $\det \tilde{I} = 0$. If we take a rotated coordinate system with a different $\tilde{I}$, then it will be related to this by a similarity transformation which will keep the value of the determinant unchanged.

For fully unpolarised light, the off-diagonal elements in the $\tilde{I}$ matrix are zero and the diagonal terms are equal; thus $\tilde{I}$ for an arbitrary light can be decomposed into a fully unpolarised and a fully polarised component.

$$\begin{aligned} \tilde{I} &= \tilde{I}^{(u)} + \tilde{I}^{(p)} \\ &= A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} B & D \\ D^* & C \end{pmatrix} \qquad BC = |D|^2 \end{aligned}$$

$B = I_{xx} - A$; $C = I_{yy} - A$. We can represent the above equation in the Cayley-Hamilton form for finding the eigenvalues: $\det(\tilde{I} - A\mathbf{I}) = 0$ where A is an eigenvalue of $\tilde{I}$. Expanding the determinant, we get

$$\begin{aligned} (I_{xx} - A)(I_{yy} - B) - |I_{xy}|^2 &= 0 \\ \Rightarrow A^2 - \operatorname{Tr} \tilde{I} A + \det \tilde{I} &= 0 \end{aligned}$$

The solution of the quadratic equation is

$$A = \frac{1}{2}\left[\mathrm{Tr}\,\tilde{I} \pm \sqrt{(\mathrm{Tr}\,\tilde{I})^2 - 4\det\tilde{I}}\right]$$

from which

$$\begin{aligned} B &= \frac{1}{2}\left[I_{xx} - I_{yy} \mp \sqrt{(\mathrm{Tr}\,\tilde{I})^2 - 4\det\tilde{I}}\right] \\ C &= \frac{1}{2}\left[I_{yy} - I_{xx} \mp \sqrt{(\mathrm{Tr}\,\tilde{I})^2 - 4\det\tilde{I}}\right] \end{aligned}$$

However B, C are positive. We shall see that only one of the two possible solutions above are positive and hence give an unique value for B and C. To see this we take the part under the radical

$$\begin{aligned} (\mathrm{Tr}\,\tilde{I})^2 - 4\det\tilde{I} &= (I_{xx} + I_{yy})^2 - 4I_{xx}I_{yy} + 4|I_{xy}|^2 \\ &= (I_{xx} - I_{yy})^2 + 4|I_{xy}|^2 \\ \Rightarrow \sqrt{(\mathrm{Tr}\,\tilde{I})^2 - 4\det\tilde{I}} &\geq |I_{xx} - I_{yy}| \end{aligned}$$

Thus to ensure the positivity of B, C we have to take the plus sign before the square root.

1.2.1. Degree of polarisation. While i_{xy} is sufficient to measure the extent of polarisation, a more widely used quantity is the *degree of polarisation* which can be directly measured by modern ellipsometers. The degree of polarisation is defined as

$$\begin{aligned} \mathrm{DoP} &\equiv \frac{\mathrm{Tr}\,\tilde{I}^{(p)}}{\mathrm{Tr}\,\tilde{I}} = \frac{\sqrt{(\mathrm{Tr}\,\tilde{I})^2 - 4\det\tilde{I}}}{\mathrm{Tr}\,\tilde{I}} \\ &= \sqrt{1 - \frac{4\det\tilde{I}}{(\mathrm{Tr}\,\tilde{I})^2}} \\ &= \sqrt{1 - \frac{I_{xx}I_{yy}(1 - i_{xy}^2)}{(I_{xx} + I_{yy})^2}} \end{aligned}$$

1.3. Mueller calculus

Now that we are mostly working with Stokes parameters, we would like a way to represent a polariser or a phase shifter in the form of a 4×4 matrix which directly operates on the corresponding Stokes vector. Such a matrix is known as a Mueller matrix. To get the Mueller matrix from the Jones matrix, we start with the intensity matrix

$$\begin{aligned}
\tilde{I}' &= \left\langle \tilde{E}^* \tilde{E}^T \right\rangle = \left\langle J^* E^* E^T J^T \right\rangle \\
&= J^* \tilde{I} J^T \\
\tilde{I}'_{ij} &= \sum_{k,l} J^*_{ik} \tilde{I}_{kl} J^T_{lj} \\
&= \sum_{k,l} J^*_{ik} J_{jl} \tilde{I}_{kl}
\end{aligned}$$

Thus $\tilde{I}$ transforms like a tensor. Essentially the Mueller matrix can be obtained from the Jones matrix by a direct product of the Jones matrices. Let's take a look at how that comes; if $T_{ij} = A_i B_j$ and $A'_i = R_{ik} A_k$; $B'_j = S_{jl} B_l$ then $T'_{ij} = R_{ik} S_{jl} T_{kl}$ and the direct product is defined as $(R \otimes S)_{ij,kl} = R_{ik} S_{jl}$.

The Stokes parameters are related to the intensity matrix as $S = A\tilde{\tilde{I}}$ where $\tilde{\tilde{I}}$ denotes the column vector corresponding to the intensity matrix and A is the matrix defined on page 4.

$$\begin{aligned}
S' &= A\tilde{\tilde{I}}' = A(J^* \otimes J)\tilde{\tilde{I}} \\
&= \underbrace{A(J^* \otimes J)A^{-1}}_{M} S
\end{aligned}$$

Here M is the Mueller matrix corresponding to the Jones matrix J. The Mueller matrix is more general than the Jones matrix and can represent transformations between partially polarised states as well. The Mueller matrix can be measured experimentally; for example by passing unpolarised light (1 0 0 0) we can find the first column of the Mueller matrix. However we can not pass (0 1 0 0) as it lies outside the Poincaré sphere. So we have to pass (1 1 0 0), (1 0 1 0) etc. to find out the other components. Using the Jones matrices for polariser, half-wave and quarter wave plates we can find out the corresponding Mueller matrices: (by convention, the polariser is assumed to be along the x axis, or $\theta = 0$)

$$\begin{array}{ccc}
\text{polariser} & \text{half-wave plate} & \text{quarter-wave plate} \\
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} &
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}$$

1.4. References

The image of the Poincaré sphere has been taken from the Wikipedia article on polarisation and was edited for the report.