Radiation from an Accelerated Charge

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Abstract

In this we calculate the radiated power from an accelerated charge which has just come to rest using geometry of electric field lines. The result applies to a wide variety of systems from radio antennas to atoms and nuclei.

1 The problem

We consider a charged particle which was moving with a velocity $v_0 \ll c$ for $t < 0$. At $t = 0$ it starts decelerating uniformly and comes to a stop in time τ. The distance covered in this time is $\frac{1}{2}v_0\tau$. We'll find out how the electric field looks like at a given time $t = T \gg \tau$. As electromagnetic waves travel at the speed of light c, observers farther away from the origin than $R = cT$ would not be aware of the particle's deceleration. Throughout this region $(R > cT)$ which we call region I, the electric field is that of a charge moving with constant velocity v_0. The field lines would appear to emanate from the position $x = v_0T$ which is the distance the particle would have travelled had it not decelerated. However for any observer who is within the spherical shell of radius $c(T - \tau)$ (in region II), the field is that of a charge at rest close to the origin at $x = \frac{1}{2}v_0\tau$ (fig 1)

2 How to join the field lines?

It is a relatively simple matter to connect up the inner and outer field lines. There is only one way to do this that is consistent with Gauss' law. We start

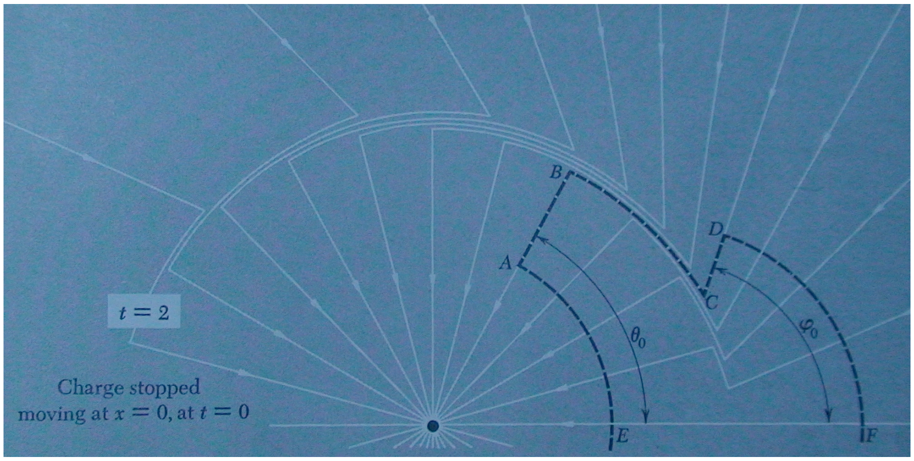

Figure 1: A charged particle brought to rest by constant deceleration starting at t=0

with some point A in region II on the radial field line making angle θ_0 with the x-axis till we reach the $R = c(T - \tau)$, follow the field line till we emerge out into region I and continue on a line making an angle ϕ_0 with the x-axis (fig 1). If we rotate EABCDF about the x axis we would generate a surface of revolution which encloses no charge. By applying Gauss' law to this surface we would derive a relationship between θ_0 and ϕ_0 . The contribution to the integral $\int \vec{E}.d\vec{S}$ would come from the caps formed by revolving AE and DF about x axis (fig 2). The flux through the inner cap is

$$\int_0^{\theta_0} \frac{q}{r^2} 2\pi r^2 \sin\theta d\theta = 2\pi q \int_0^{\theta_0} \sin\theta d\theta \tag{1}$$

and through the outer cap

$$\int_0^{\phi_0} \frac{q}{r^2} \frac{1-\beta^2}{(1-\beta^2\sin^2\phi)^{3/2}} 2\pi r^2 \sin\phi d\phi = 2\pi q \int_0^{\phi_0} \frac{1-\beta^2}{(1-\beta^2\sin^2\phi)^{3/2}} \sin\phi d\phi \tag{2}$$

where we've used the form of the electric field for a uniformly moving charged particle viz.

$$E = \frac{q}{r^2} \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \tag{3}$$

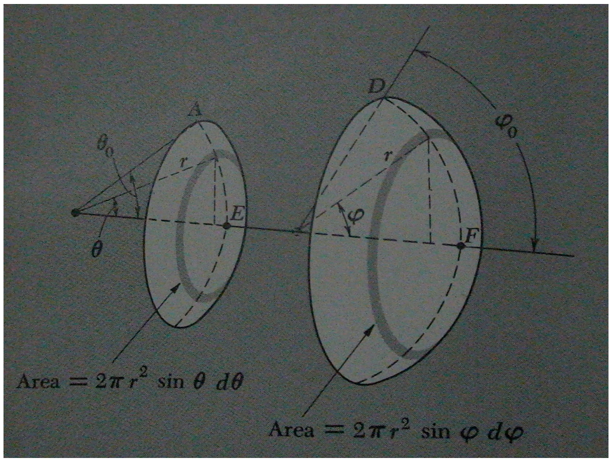

Figure 2: Only the inner and the outer cap contribute to the surface integral of $\vec{E}$

where r and θ are measured relative to the position of the charge. The flux into the inner cap is the flux out through the outer cap, and hence

$$\int_0^{\theta_0} \sin\theta d\theta = \int_0^{\phi_0} \frac{1-\beta^2}{(1-\beta^2\sin^2\phi)^{3/2}} \sin\phi d\phi \tag{4}$$

which simplifies to

$$\cos\theta_0 = \frac{\cos\phi_0}{\sqrt{1-\beta^2\sin^2\phi_0}} \tag{5}$$

and equivalently to

$$\tan\phi_0 = \gamma\tan\theta_0 \tag{6}$$

This reduces to $\theta_0 = \phi_0$ under the condition $\gamma \to 1$ i.e., for non-relativistic speeds.

3 Using the geometry of the electric field lines

The only problem lies in finding the electric field in the annular region between $R = c(T-\tau)$ and $R = cT$. We use Gauss' law. We have already seen

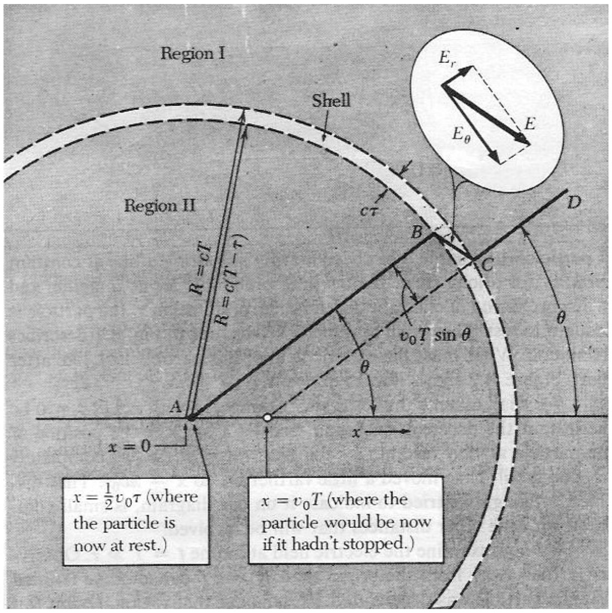

Figure 3: Space diagram for $t = T \gg \tau$

that AB and CD must be part of the same field line connected by BC. Also we have $\theta_0 = \phi_0 = \theta$ since $v_0 \ll c$. The electric field within the shell which has both radial and transverse components. From the geometry of the field lines (fig 3) in this annular region

$$\frac{E_\theta}{E_r} = \frac{v_0 T \sin\theta}{c\tau} \tag{7}$$

Taking a small Gaussian pillbox near B, we see that E_r has the same value within the shell thickness as in region II at B. Thus $E_r = q/R^2 = q/c^2T^2$. Substituting, we get

$$E_\theta = \frac{v_0 T \sin\theta}{c\tau} E_r = \frac{q v_0 \sin\theta}{c^3 T \tau} \tag{8}$$

Rewriting using the results $v_0/\tau = a$ and $R = cT$

$$E_\theta = \frac{q a \sin\theta}{c^2 R} \tag{9}$$

It is remarkable that E_θ in contrast to E_r is proportional to $1/R$ instead of $1/R^2$. With increasing time and thus increasing R, the transverse field will thus become much stronger than the radial field. Along with this transverse electric field, there will also be a magnetic field orthogonal to $\mathbf{E}$ which can be found out by applying Ampere's law in the integral form.

4 Power radiated from an accelerating charge

Now we shall find the energy stored in the electric field in the whole spherical shell. The energy density for the transverse field is (we are ignoring the radial component as it can be neglected in comparison to the transverse for large R)

$$\frac{E_\theta^2}{8\pi} = \frac{q^2 a^2 \sin^2\theta}{8\pi R^2 c^4} \tag{10}$$

The volume of the shell is $4\pi R^2 c\tau$ and the average value of $\sin^2\theta$ over the whole sphere is $2/3$. We can see this by taking the polar axis as the x axis ($x^2 = R^2\cos^2\theta$). Due to symmetry, $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = R^2/3$. Thus $\langle\cos^2\theta\rangle = 1/3$, implying $\langle\sin^2\theta\rangle = 2/3$. Then the total energy of the transverse electric field is

$$\frac{2}{3} 4\pi R^2 c\tau \frac{q^2 a^2}{8\pi R^2 c^4} = \frac{q^2 a^2 \tau}{3c^3} \tag{11}$$

We add an equal amount of energy stored in the magnetic field to get the total energy $2q^2a^2\tau/3c^3$. As there is no dependence on R, this energy simply travels out undiminished with the speed of light c. An observer would observe the radiation as a pulse with the time width τ. Thus the power radiated during the accelration process is

$$P_{rad} = \frac{2}{3}\frac{q^2 a^2}{c^3} \tag{12}$$

This is a Lorentz invariant quantity since P_{rad} is energy/time, and energy transforms like time, each being the fourth component of a four-vector. This expression gives the instantaneous rate of radiation by a charged particle moving with variable acceleration.

References

[1] Purcell E M 1981 *Electricity and Magnetism*, Berkeley Physics Course vol 2 (Asian Students edition) (Singapore: McGraw-Hill)