PH4208: Nonlinear Dynamics

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Chapter 1

Lecture notes of 7th January 2014

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1.1 Bifurcation

In the previous class, dependence of the dynamics of the system on initial conditions was studied. Geometric approach helped us in extracting its qualitative features for different types of initial conditions. But a system's dynamics not only depends on initial conditions specified. It also depends upon *parameters*. They are variables that characterize the system (and therefore, are part of governing differential or difference equation) and stay constant during system's time-evolution. The qualitative structure of flow can change as parameters are varied. In this aspect, even one-dimensional systems, which don't exhibit any oscillations, offers valuable lessons. We would find that fixed points can be created or destroyed, or their stabilities can change as parameters are varied.

The qualitative changes in the dynamics as parameters are changed are called **bifurcations**, and the parameter values at which they occur are called **bifurcation points**.

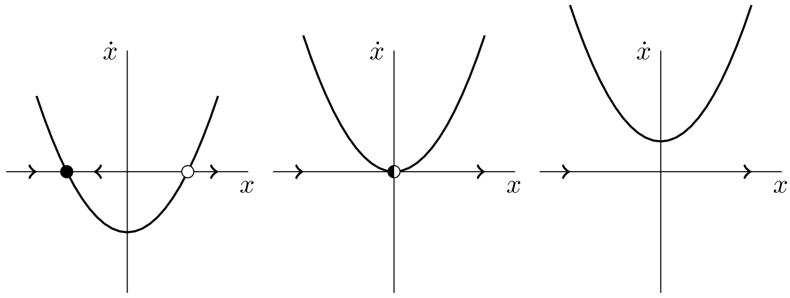
1.1.1 Saddle-node Bifurcation

If you are looking for a mechanism by which fixed points are created and destroyed, saddle-node bifurcation would do it for you. Its two prototypical examples are

$$\dot{x} = r + x^2, \quad \dot{x} = r - x^2 \tag{1.1}$$

where r is a parameter, which may be positive, negative or zero.

For the first equation, when r is negative, there are two fixed points, one stable and one unstable. For $r = 0$, there is one half-stable fixed point, and for $r > 0$, there is no fixed point. Moreover, for $r \leq 0$, we note that fixed point x^* satisfies $r + x^2$. We can depict this graphically as follows.



$(a) r < 0$ $(b) r = 0$ $(c) r > 0$

Figure 1.1: fixed points for different r

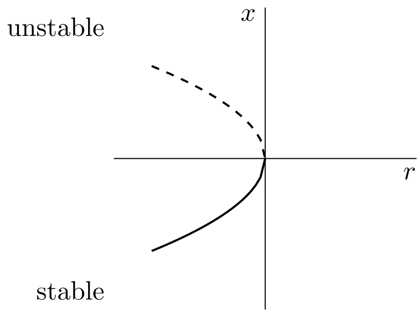


Figure 1.2: Bifurcation diagram for $\dot{x} = r + x^2$

Similarly, for the second equation, we can see that it has no fixed points for $r < 0$, one half-stable fixed point for $r = 0$, and two fixed points(one stable and one unstable) for $r > 0$. Moreover, for $r \geq 0$ we note that fixed point x^* satisfies $r - x^2 = 0$. We can depict this graphically as follows.

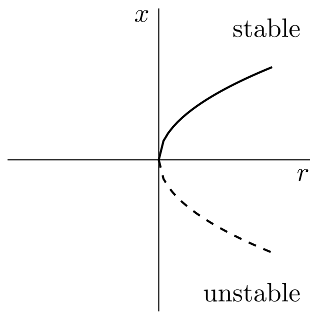
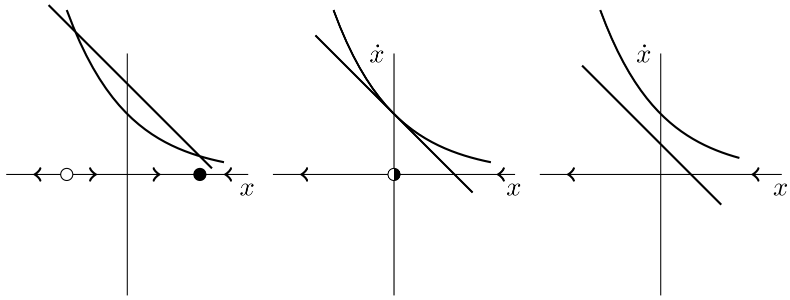


Figure 1.3: Bifurcation diagram for $\dot{x} = r - x^2$

There are various name for saddle-node bifurcation– fold bifurcation, turning-point bifurcation and blue sky bifurcation.



$(a) r > 1$ $(b) r = 1$ $(c) r < 1$

Figure 1.4: Geomterical method for finding fixed points for $\dot{x} = r - x - e^{-x}$

Example

$$\dot{x} = r - x - e^{-x} \tag{1.2}$$

Using geometric approach, one finds that for $r < r_c$, there are no fixed points and two for $r > r_c$, where r_c is the bifurcation point. For $r = r_c$, there is one half-stable fixed point. Using tangency condition, we find that $r_c = 1$.

Normal forms

Comparing equation (1.2) with our protypical examples, we suspect that it's the old wine in the new bottle. Indeed, this is the case as the following calculation shows.

$$\begin{aligned} \dot{x} &= r - x - e^{-x} \\ &= r - x - [1 - x + \frac{x^2}{2} + \cdots] \\ &\backsimeq r - 1 - \frac{x^2}{2}, \quad x \ll 1 \end{aligned} \tag{1.3}$$

The change of variables can recast the above equation into our old friend $\dot{X} = R - X^2$, where $X = \alpha x$. On solving for α and R, we get $r - 1 = 2R$ and $x = 2X$.

The above example shows why we say the equations $\dot{x} = r + x^2$ or $\dot{x} = r - x^2$ are "prototypical". In a certain sense, they are representative of *all* saddle-node bifurcations. Conventionally, they are known as **normal forms** for the saddle node-bifurcation. In general, recipe for finding the bifurcation point(r_c) would be to solve for r_c satisfying

- $f(x_c, r_c) = 0$
- $\frac{\partial f}{\partial r}(x_c, r_c) = 0$

where $x_c = x^*\Big|_{r=r_c}$.

Using Taylor expansion, one can see that why saddle node-bifurcations typically have these algebraic forms. Let's examine the behavior of $\dot{x} = f(x, r)$ near the bifurcation point (x_c, r_c).

$$\begin{aligned}
\dot{x} &= f(x, r) \\
&= f(r_c, x_c) + (x - x_c)\frac{\partial f}{\partial x}\bigg|_{x_c, r_c} + (r - r_c)\frac{\partial f}{\partial r}\bigg|_{x_c, r_c} + (x - x_c)^2 \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{x_c, r_c} + \cdots \\
&= (r - r_c)\frac{\partial f}{\partial r}\bigg|_{x_c, r_c} + (x - x_c)^2 \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{x_c, r_c} + \cdots \\
&\simeq a(r - r_c) + b(x - x_c)^2, \quad (x - x_c) \ll 1
\end{aligned}$$

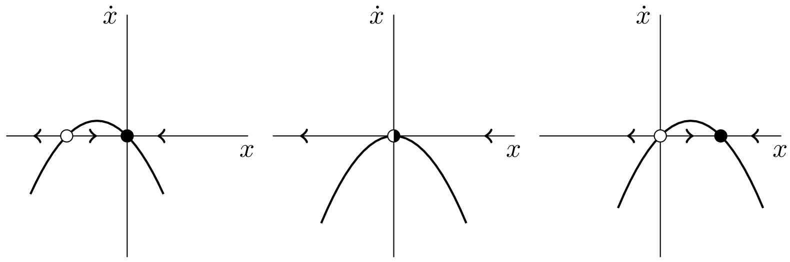
where a=$\frac{\partial f}{\partial r}\bigg|_{x_c, r_c}$, b=$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{x_c, r_c}$

In the above, we have used the conditions for bifurcation point.

1.1.2 Transcritical Bifurcation

Here the number of fixed points remains fixed as the parameter is changed but their nature might change. The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2 \tag{1.4}$$



$(a) r < 0$ $\quad$ $(b) r = 0$ $\quad$ $(c) r > 0$

Figure 1.5: Fixed points for different r

Zero remains a fixed point for *all* values of r. For $r < 0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. For $r = 0$, there is no fixed point except the one half-stable sitting at origin. For $r > 0$, the origin has become unstable, and $x^* = r$ is now stable. We note that fixed point (x^*) satisfies $x = 0$ and $x = r$ for all values of r. Thus, graphically, we get figure(1.6)

We obtain similar diagram for first-order phase transition plot of gibbs free energy per unit mass (g) as a function of temperature. We find that there is a kink in the diagram. The underlying physics can be explained using Transcritical Bifurcation. Liquid has a lower entropy than gas. Since, liquid's g rises above than that of gas after transition temperature(T_c), substance transits from liquid state to gaseous. The gaseous state becomes more stable after T_c due to lower g. (Note that g remains continous)

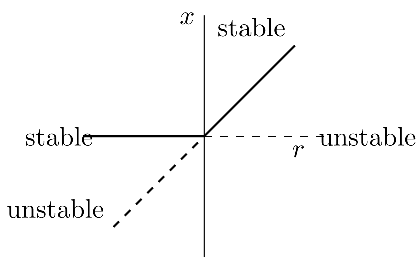


Figure 1.6: Bifurcation diagram for Transcritical Bifurcation

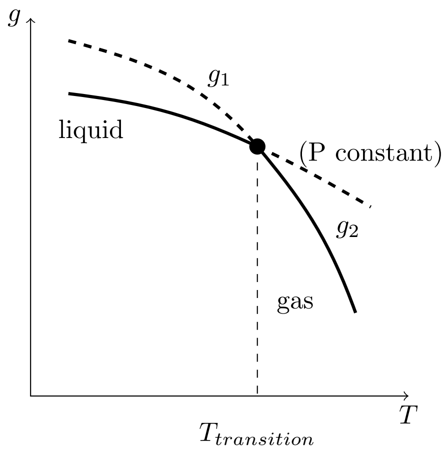


Figure 1.7: First order phase transition

1.1.3 Pitchfork Bifurcation

Similar to transcritical bifurcation, here also a point remains fixed as parameter is varied. But what sets it apart is that new fixed points appear suddenly in addition to the *fixed* fixed point.

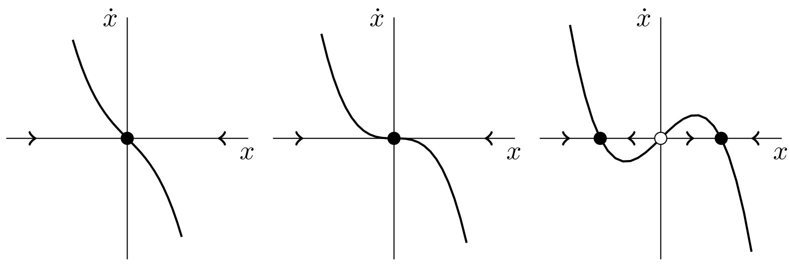
Supercritical Pitchfork Bifurcation

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3 \tag{1.5}$$

Now since, the above equation is symmetric under $x \to -x$, the solutions would be odd as plotted below for different values of r.

For all values of r, origin is the fixed point. For $r < 0$, it is the only fixed point, and it is stable. When $r = 0$, it is still stable but the derivative goes to zero at origin. As a result, the decay is much slower. **Critical slowing down** in physics literature shows the same phenomenon. For $r > 0$, origin becomes unstable, which is more than compensated by appearance of two stable fixed points on either sides of origin. We also note that fixed point x^* satisfies $x = 0$ for all values of r. For $r > 0$, it *also* satisfies $x = \pm\sqrt{r}$. Thus, graphically we obtain



$(a) r < 0$ $(b) r = 0$ $(c) r > 0$

Figure 1.8: fixed points for different r

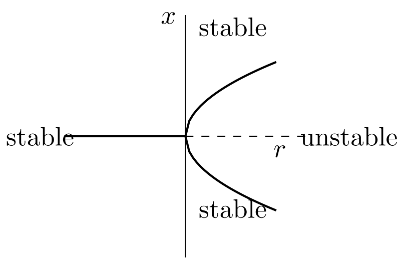


Figure 1.9: Bifurcation diagram for Supercritical Bifurcation

1.2 Uniqueness of solution

Let's try to find the solution to $\dot{x} = x^n, n < 1$ starting from $x_0 = 0$. Newly learned geometric approach tells us that the solution is trivially x(t)=0 as x=0 is a fixed point. If you are skeptic enough not to accept it, and want to see it for yourself using analytic method, you would be rewarded.

$$\begin{aligned}\int x^{-n} dx &= \int dt \\ \frac{x^{1-n}}{1-n} &= t, \quad \because x(0) = 0, n < 1 \\ x &= [(1-n)t]^{1/1-n} \end{aligned} \tag{1.6}$$

So, we find that there are *at least* two solutions for the given initial condition! The uniqueness of trajectories in the phase space (which was argued earlier as the reason for promoting time as one of the dimensions of the phase-space for non-autonomous systems) has broken down.

To deal with such pathological cases, we ask for help from Picard's Theorem, which says an initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1.7}$$

it would have unique solution on some time interval $(-\tau, \tau)$ about t=0, if $f'(x)$ is continuous on an open interval R of the x-axis, and $x_0 \in R$

In the above example, derivative of $f(x) = x^n, n < 1$ is not continuous at $x = 0$, and thus, uniqueness is not guaranteed.

Now once we have the recipe for identifying the problematic cases, how do we understand them? As pointed out by Strogatz, we would find that non-uniqueness is sometimes not that bad. It may be obvious and natural in some cases. He talks about a leaky bucket. Physics can uniquely predict how much time it takes for the bucket to empty given the initial level of water in it. However, by seeing a puddle beneath such an empty bucket, it is not possible for even Sherlock Holmes to tell when the bucket was full. For all you know, Strogatz says, it could have emptied a minute ago, ten minutes ago, or whatever. (Here we are ignoring evaporation rate.)

Bibliography

[1] Kerson Huang. *Statiscal Mechanics*. Wiley, 2000.

[2] Steven Strogatz. Nonlinear dynamics and chaos: with applications to physics, biology, chemistry and engineering. 2001.