PH4105/5105 Problem set 1

Q 1) Explicitly list all possible topologies of a three element set $\{a, b, c\}$ (Note : there are precisely 29 of them!)

Q 2) Consider the set $\mathbb{R}^2$ with a "distance" $d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_\infty (x, y) \equiv \max \{|x_1 - y_1|, |x_2 - y_2|\}$$

Sketch the open balls $B_x^{d_\infty}(r)$ under this metric. Show that these open balls are open sets under the "standard", *i.e.* d_2 metric. Show conversely that the open balls in the d_2 metric are open sets under d_∞. Hence argue that they both lead to the same topology.

Q 3) A topological space $(X, \mathcal{T})$ is called Hausdorff if $\forall x, y \in X$, either $x = y$ or $\exists U_x, U_y \in \mathcal{T}$ such that $x \in U_x$, $y \in U_y$ and $U_x \cup U_y = \emptyset$.

(a) Which of the 29 topologies that you listed in the answer to question 1 are Hausdorff?

(b) Prove that the metric topology that is induced on a metric space by the open balls is Hausdorff.

Q 4) We define the Euclidean distance d_2 on $\mathbb{R}^n$ by

$$d_2 (x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Prove that this is a metric.

Q 5) Given a Metric space (M, d) prove that the function defined by

$$\forall x, y \in M, \qquad \overline{d}(x, y) \equiv \frac{d(x, y)}{1 + d(x, y)}$$

is also a valid metric.

Comapre the metric topologies on (M, d) and $\left(M, \overline{d}\right)$.