PH4105 Problem Set 2

Q 1) Consider the following set of charts (U_i, ϕ_i) on the sphere

$$S^2 \equiv \left\{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1\right\}$$

(i) $U_1 = \{(x, y, z) \in S^2 \,:\, z > 0\}$, $\phi_1(x, y, z) = (x, y)$

(ii) $U_2 = \{(x, y, z) \in S^2 \,:\, z < 0\}$, $\phi_2(x, y, z) = (x, y)$

(iii) $U_3 = \{(x, y, z) \in S^2 \,:\, y > 0\}$, $\phi_3(x, y, z) = (x, z)$

(iv) $U_4 = \{(x, y, z) \in S^2 \,:\, y < 0\}$, $\phi_4(x, y, z) = (x, z)$

(v) $U_5 = \{(x, y, z) \in S^2 \,:\, x > 0\}$, $\phi_5(x, y, z) = (y, z)$

(vi) $U_6 = \{(x, y, z) \in S^2 \,:\, x < 0\}$, $\phi_6(x, y, z) = (y, z)$

Verify that each one is a bona fide chart on S^2. Do these charts form a C^∞ subatlas on S^2?

Q 2) a) Consider the chart (U_1, ϕ_1) in the example above and also consider the chart (V, χ) where χ maps a point on the sphere to the corresponding spherical polar coordinates θ, φ . What is the largest possible domain V for which (V, χ) is a chart? Write down explicit forms for the transition functions $\phi_1 \circ \chi^{-1}$ and $\chi \circ \phi_1^{-1}$ as well as their corresponding domains and ranges.

b) A function f on the sphere is defined to be $\frac{d^3}{1+d^2}$ where d is the distance of the point from the equatorial plane. Write down the proxy functions $f \circ \phi_1^{-1}$ and $f \circ \chi^{-1}$ for this. What is the value of this function at the point $p = \phi_1^{-1}(0.36, 0.48)$? Check that applying the two proxy functions to $\phi_1(p)$ and $\chi(p)$, respectively, returns the same value.

c) Express the basis vectors $\left(\frac{\partial}{\partial \theta}\right)_p$ and $\left(\frac{\partial}{\partial \varphi}\right)_p$ in terms of the basis vectors $\left(\frac{\partial}{\partial x}\right)_p$ and $\left(\frac{\partial}{\partial y}\right)_p$ where $p = \phi_1^{-1}(0.36, 0.48)$.

d) A tangent vector at this point is above is denoted by $v_p = 2\left(\frac{\partial}{\partial x}\right)_p - 3\left(\frac{\partial}{\partial y}\right)_p$. Find the value of $v_p\left[f\right]$ where f is the function defined in part (b) above.

e) Consider a vector field defined in U_1 by

$$X\left(p\right) = \frac{x^2}{x^2+y^2}\left(\frac{\partial}{\partial x}\right)_p - \frac{3y^2}{x^2-y^2}\left(\frac{\partial}{\partial y}\right)_p$$

What is the function Xf where f is the function defined in part (b) above?

f) Find Yf where Y is defined in V by

$$X\left(p\right) = \sin^2\theta\cos^2\varphi\left(\frac{\partial}{\partial\theta}\right)_p - \cos^2\theta\sin^2\varphi\left(\frac{\partial}{\partial\varphi}\right)_p$$

Q 3) Write down the transition functions for the two charts defined by stereographic projections from the North and the South pole, respectively, to the equatorial plane and hence show that the corresponding charts form a C^∞ subatlas on S^2. If (ξ, η) and (ξ', η') are the respective coordinate functions, express the basis vector fields $\left(\frac{\partial}{\partial\xi}\right)$ and $\left(\frac{\partial}{\partial\eta}\right)$ in terms of $\left(\frac{\partial}{\partial\xi'}\right)$ and $\left(\frac{\partial}{\partial\eta'}\right)$.