PH4105/5105 problem set 3

Q 1) Find the integral curves starting at (x_0, y_0) and the one-parameter group of transformations generated by the vector field

$$\boldsymbol{X} = \frac{1}{x^2+y^2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

Q 2) In this problem, $\boldsymbol{X}$, $\boldsymbol{Y}, \boldsymbol{Y}_i \in \mathfrak{X}(M)$. For a rank k cotensor field t, we know that

$$\pounds_{\boldsymbol{X}} t\left(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k\right) = \boldsymbol{X}\left(t\left(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k\right)\right) - \sum_{i=1}^{k} t\left(\boldsymbol{Y}_1, \ldots, \left[\boldsymbol{X}, \boldsymbol{Y}_i\right], \ldots, \boldsymbol{Y}_k\right)$$

Using this, show that

$$\left[\pounds_{\boldsymbol{X}}, \pounds_{\boldsymbol{Y}}\right] t = \pounds_{[\boldsymbol{X},\boldsymbol{Y}]} t$$

Q 3) Remember that the exterior derivative of a k-form ω is defined by

$$\begin{aligned}(k+1)\, d\omega\left(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{k+1}\right) &= \sum_{i=1}^{k+1} (-1)^{i+1}\, \boldsymbol{X}_i\left(\omega\left(\boldsymbol{X}_1, \ldots, \hat{\boldsymbol{X}}_i, \ldots \boldsymbol{X}_{k+1}\right)\right) \\ &\quad + \sum_{1\le i<j\le k+1} \omega\left(\left[\boldsymbol{X}_i, \boldsymbol{X}_j\right] \boldsymbol{X}_1, \ldots, \hat{\boldsymbol{X}}_i, \ldots, \hat{\boldsymbol{X}}_j, \ldots \boldsymbol{X}_{k+1}\right)\end{aligned}$$

From this definition, show that $d\omega$ is multilinear and completely antisymmetric.

Q 4) If $\alpha \in \Lambda^1(M)$ and $\omega \in \Lambda^k(M)$ prove that for $X_i \in \mathfrak{X}(M)$

$$\alpha \wedge \omega\left(X_1, \ldots, X_{k+1}\right) = \frac{1}{k+1}\sum_{i=1}^{k+1} (-1)^{i+1}\, \alpha\left(X_i\right) \omega\left(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}\right)$$

Q 5) Prove that for $f \in C^\infty(M)$ and $\omega \in \Lambda^k(M)$

$$d(f\omega) = df \wedge \omega + f d\omega$$

Q 6) For $\boldsymbol{X} \in \mathfrak{X}(M)$, $\omega \in \Lambda^k(M)$, show that

$$\pounds_{\boldsymbol{X}}\omega = \boldsymbol{X} \lrcorner d\omega + d(\boldsymbol{X} \lrcorner \omega)$$

Q 7) Verify, by explicit calculation starting from the definition of the exterior derivative that

$$d^2\omega = 0$$

where $\omega \in \Lambda^2(M)$.