Chapter 6

Propagation of errors

6.1 Error in addition of two measured values

Often the value that we need from an experiment is obtained as a combination of a few measured values. You may have measured the values of two variables x and y, and may need to obtain a third variable $z = x + y$. The two measured variables have independent standard errors δx and δy. The mean of z can be easily obtained as

$$\bar{z} = \bar{x} + \bar{y}.$$

But what would be the estimate of error in $\bar{z}$?

Since the two errors are independently random, they might add to each other (Fig. 6.1(a)), they might oppose each other (Fig 6.1(b)), and all the intermidiate possibilities would occur. Statistical analysis shows that the right value is obtained by considering the errors as vectors placed in quadrature (Fig. 1(c)).

Thus, the estimated error in z is

$$\delta z = \sqrt{\delta x^2 + \delta y^2} \tag{6.1}$$

If the desired quantity z is $x - y$, then the mean value of z is obtained from the mean values of the measured variables:

$$\bar{z} = \bar{x} - \bar{y}.$$

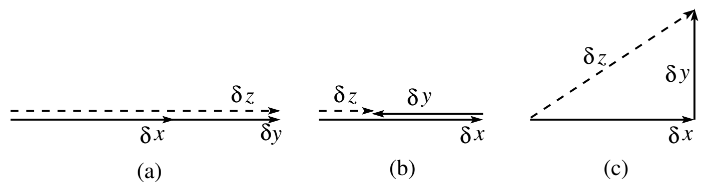

Figure 6.1: Addition of variables with errors: (a) constructive addition, (b) destructive addition, and (c) the mean.

But the errors will add up or subtract in the same way as shown in Fig. 6.1, and so the standard error in z is still given by the expression (6.1). Notice that a subtraction of two measured values would increase the percentage error in z.

6.2 Error in product of two measured values

What if the desired quantity, z, is a product of x and y: the two quantities being measured? Then we reduce it to addition form by taking logarithm:

$$\begin{aligned} z &= x \cdot y \\ \ln z &= \ln x + \ln y \end{aligned}$$

What will be the error in $\ln z$? Since it is obtained as a sum, by (6.1) we get

$$\delta \ln z = \sqrt{(\delta \ln x)^2 + (\delta \ln y)^2}$$

Now, we know

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

Expressing it in terms of incremental quantities, we get

$$\delta \ln x = \frac{\delta x}{x}$$

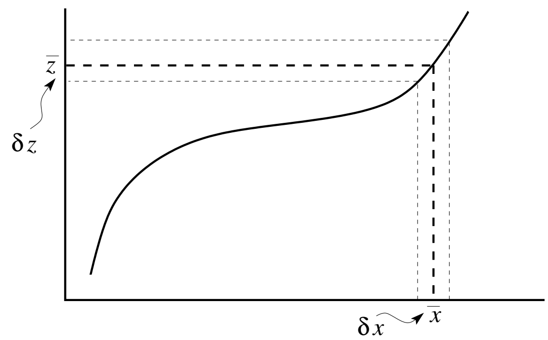

Figure 6.2: Propagation of error when z is a function of x.

Substituting, we get

$$\frac{\delta z}{z} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2} \tag{6.2}$$

Thus, if two measured quantities are multiplied, the percentage error in z is obtained in terms of the *percentage errors* of x and y.

6.3 Error in some function of a measured value

Now suppose you have measured a quantity x with a mean value $\bar{x}$ and standard error δx. The variable z is some function of x, as depicted in Fig. 6.2. From the figure it is clear that the error will propagate from x to z according to the derivative of z evaluated at $\bar{x}$, i.e.,

$$\delta z = \left|\frac{dz}{dx}\right|_{\bar{x}} \delta x.$$

If z is a function of more than one variable, say x and y, both of which are measured with some error bar, then the errors will propagate according to the partial derivatives:

$$\delta z = \sqrt{\left(\frac{\partial z}{\partial x}\delta x\right)^2 + \left(\frac{\partial z}{\partial y}\delta y\right)^2}.$$

With these basic formulae, you will be able to obtain the standard error of any variable from measurements of related variables.

Example 6.1: From the data in Example 4.3 , calculate the estimated values of the quantities $x+y$, xy, and $\sin\frac{1}{x}$, including the error bars.

Solution:

$$
\begin{aligned}
\text{Mean of } x+y &= 5.018+3.335=8.353 \\
\text{Standard Error of } x+y &= \sqrt{0.032^2+0.042^2}=0.053 \\
\text{Mean of } xy &= 5.018\times 3.335=16.735 \\
\text{Standard Error of } xy &= 16.735\times\sqrt{\left(\frac{0.032}{5.018}\right)^2+\left(\frac{0.042}{3.335}\right)^2}=0.236 \\
\text{Mean of } \sin\frac{1}{x} &= \sin\frac{1}{5.018}=0.0035
\end{aligned}
$$

Now let $z=\sin\frac{1}{x}$. Therefore

$$\frac{dz}{dx}=\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)$$

Therefore the error in z will be

$$
\begin{aligned}
\delta z &= \left|-\frac{1}{x^2}\cos\frac{1}{x}\right|\delta x \\
&= \frac{1}{5.018^2}\cos\frac{1}{5.018}\times 0.032=0.0012
\end{aligned}
$$

Hence the three quantities are to be specified, including the error bars, as

$$
\begin{aligned}
x+y &= 8.353\pm 0.053 \\
xy &= 16.735\pm 0.236 \\
\sin\frac{1}{x} &= 0.0035\pm 0.0012
\end{aligned}
$$

□

Exercise

1. Suppose that upon repeated measurements 16 times, two quantities x and y are found to have values $\bar{x} = 2.57$ with standard deviation 0.6 and $\bar{y} = 3.29$ with standard deviation 0.9. How will you specify the quantities $x + y$, $x - y$, xy and x/y?