Cut Locus of Submanifolds: A Geometric Viewpoint

Seminar GANIT
IIT Gandhinagar

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IISER KOLKATA

Outline of the talk

Background

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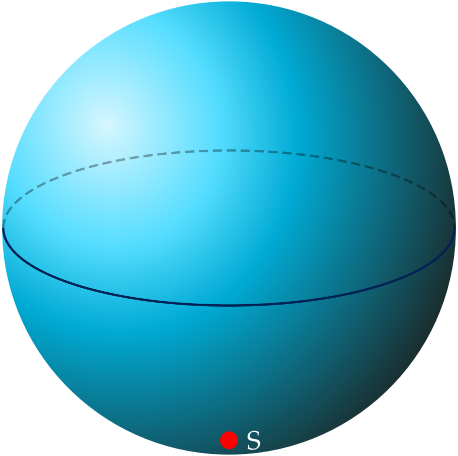

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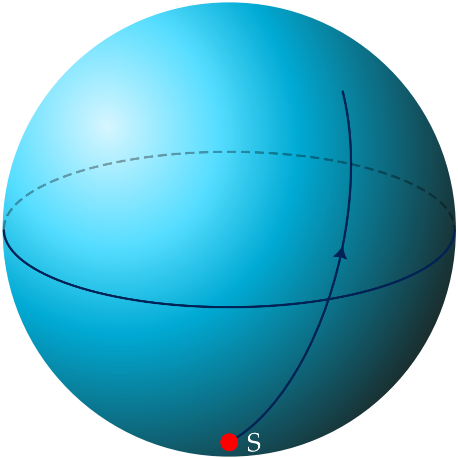

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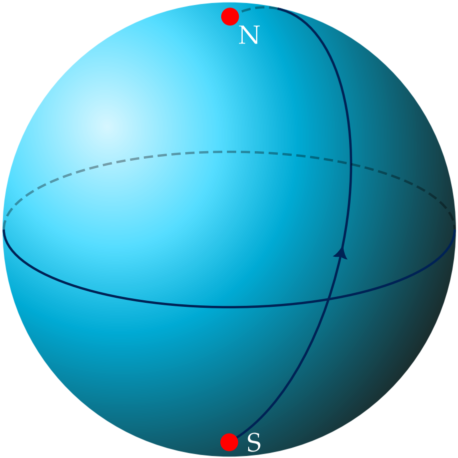

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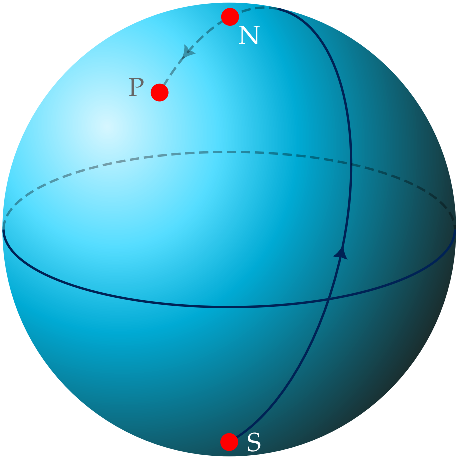

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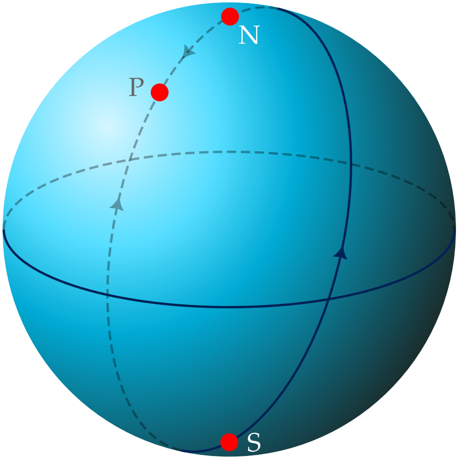

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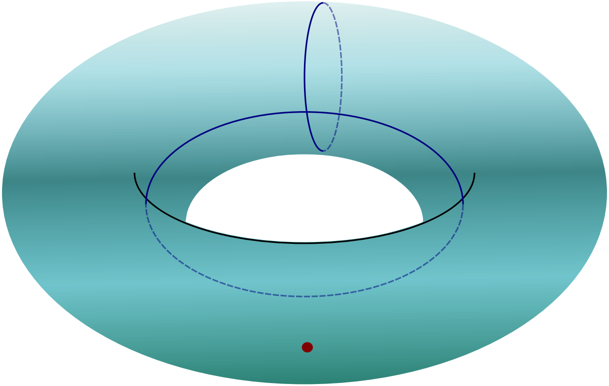

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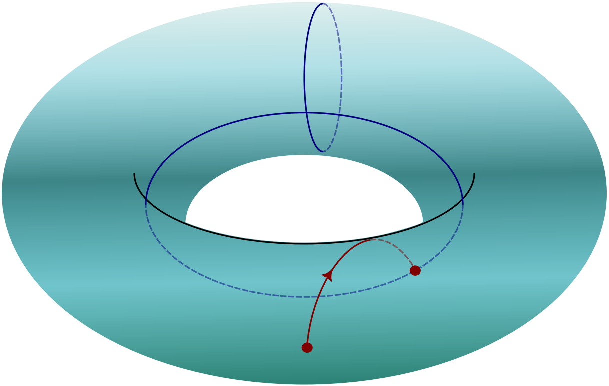

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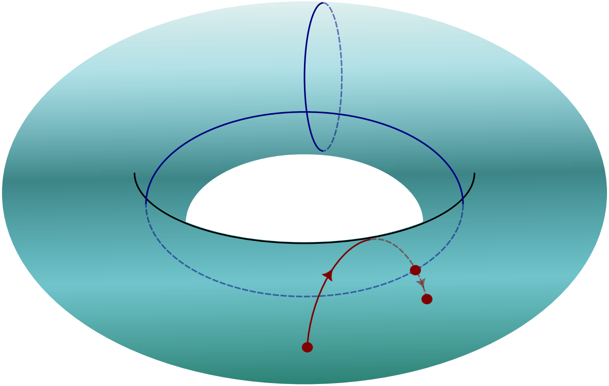

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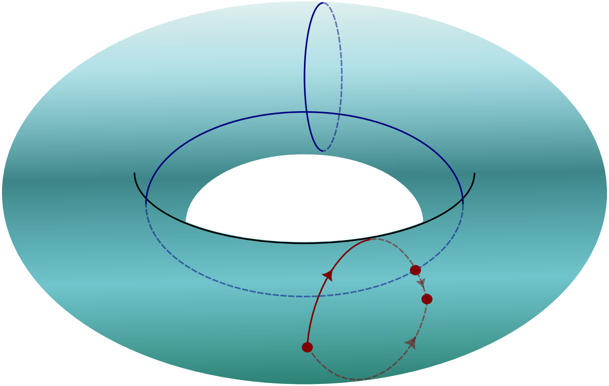

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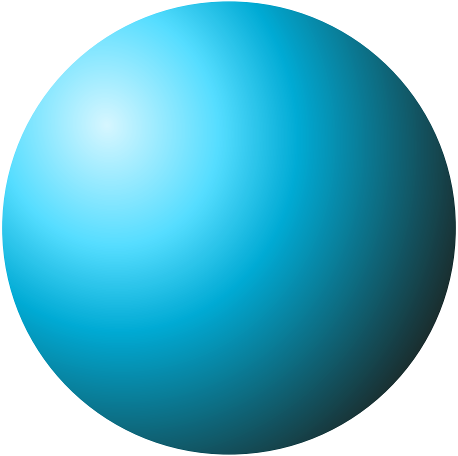

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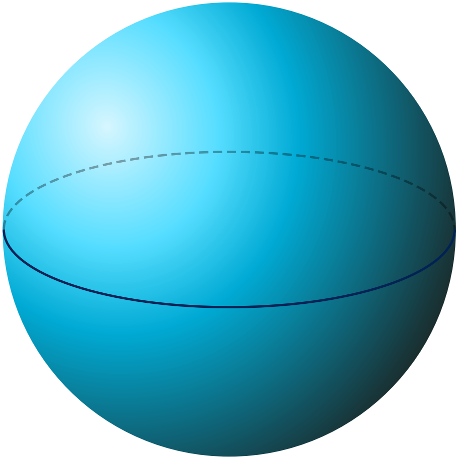

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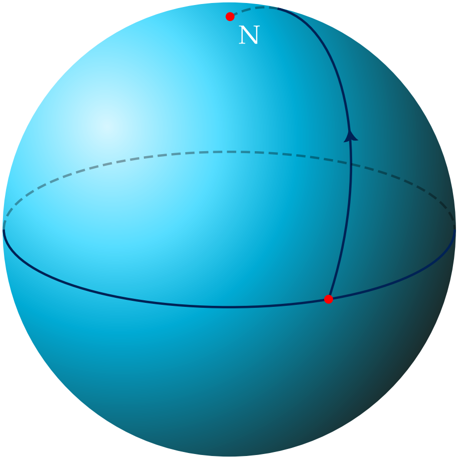

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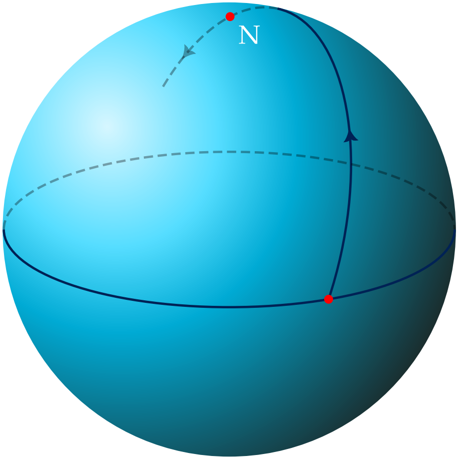

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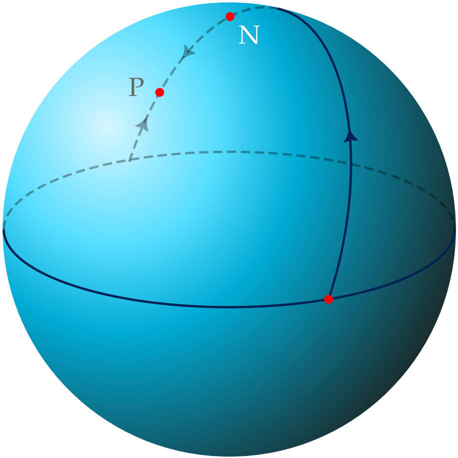

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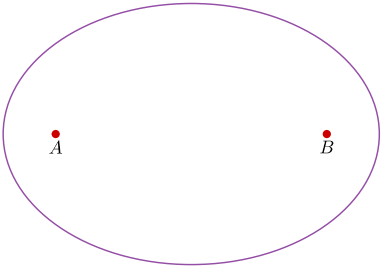

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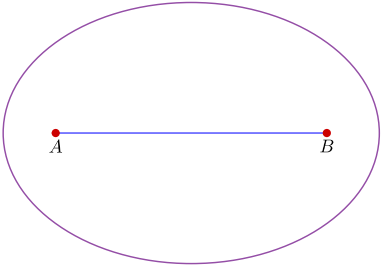

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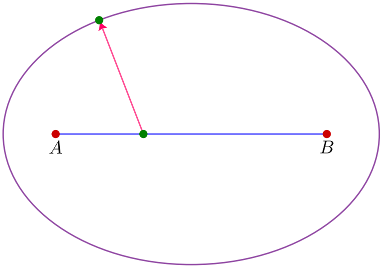

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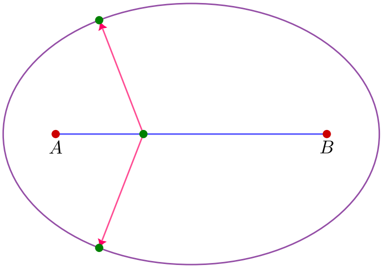

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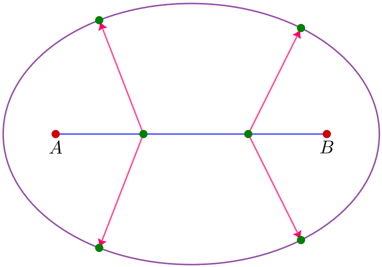

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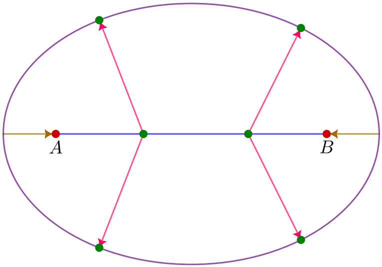

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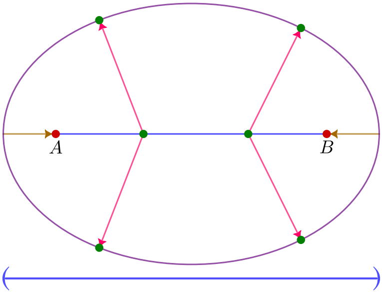

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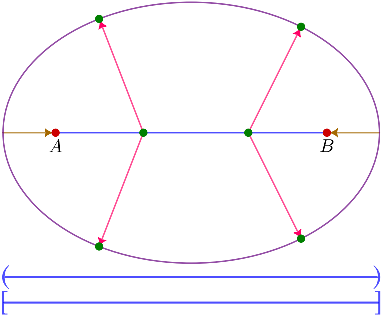

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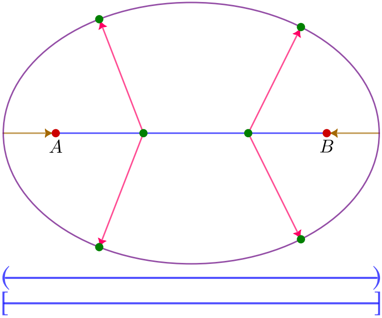

Theorem (Basu, S. and Prasad, S. [1])

For a complete Riemannian manifold M and a compact submanifold N of M,

$$\overline{\mathrm{Se}(N)} = \mathrm{Cu}(N).$$

Deformation of complement of the cut locus

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The cut locus $\mathrm{Cu}(N)$ is a strong deformation retract of $M - N$. In particular, $(M, \mathrm{Cu}(N))$ is a good pair and the number of path components of $\mathrm{Cu}(N)$ equals that of $M - N$.

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$$\mathbf{s} : S(\nu) \to [0,\infty],\ \mathbf{s}(v) := \sup\{t \in [0,\infty) \mid \gamma_v|_{[0,t]} \text{ is an } N\text{-geodesic}\},$$

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$$U_0(N) := \{av : 0 \le a < \mathbf{s}(v),\ v \in S(\nu)\}.$$

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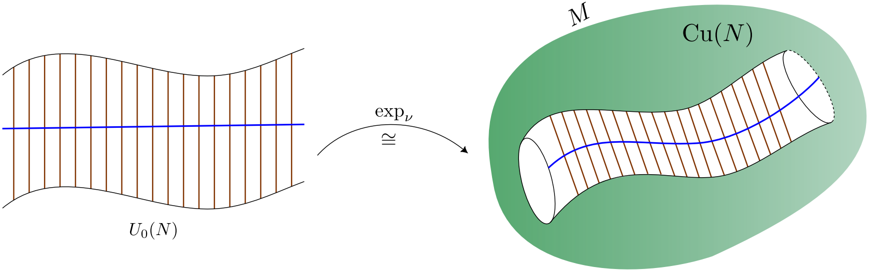
Note that $\exp_\nu$ is a diffeomorphism on $U_0(N)$ and set $U(N) = \exp_\nu(U_0(N)) = M - \mathrm{Cu}(N)$.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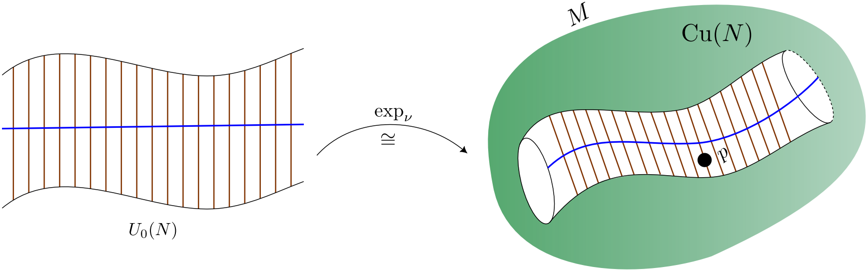
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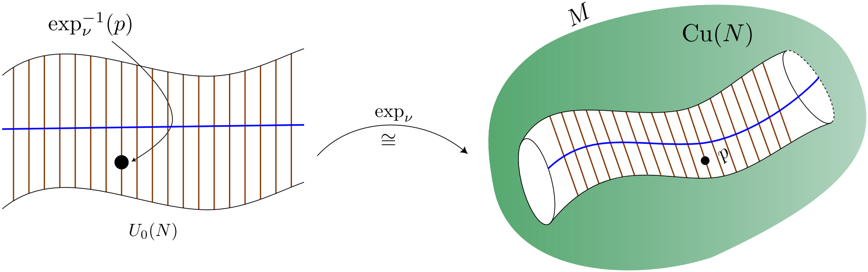

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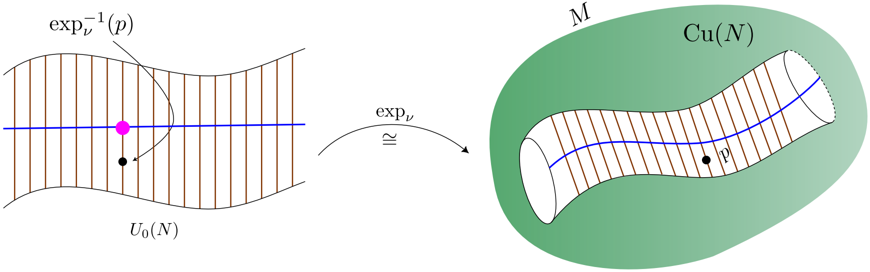
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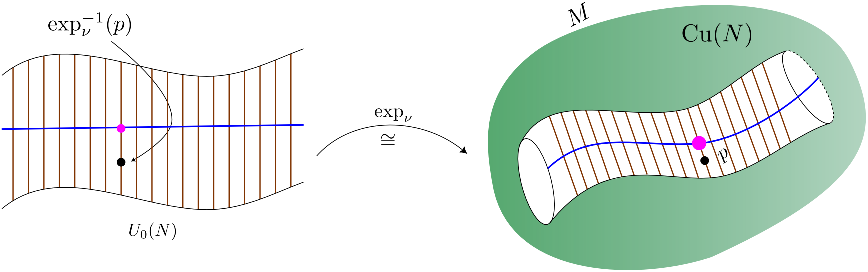
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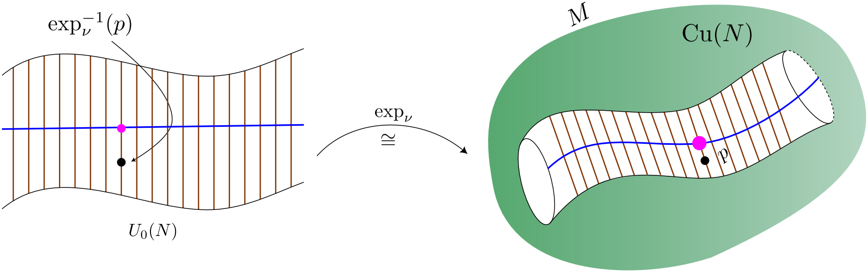

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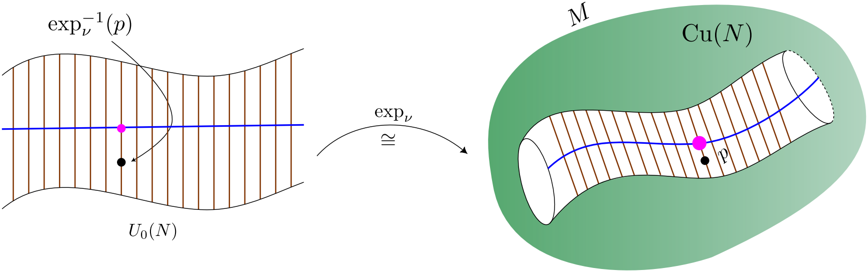

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$$H : U_0(N) \times [0,1] \to U_0(N), ((p,av),t) \mapsto (p,tav).$$

Now consider the following diagram:

$$\begin{array}{ccc} U_0(N)\times[0,1] & \xrightarrow{H} & U_0(N) \\ \uparrow{\scriptstyle \exp_v^{-1}} & & \downarrow{\scriptstyle exp_v} \\ U\times[0,1] & \xrightarrow{F} & U\cong M-\mathrm{Cu}(N) \end{array}$$

Now consider the following diagram:

$$\begin{array}{ccc} U_0(N) \times [0,1] & \xrightarrow{\ H\ } & U_0(N) \\ \uparrow{\scriptstyle \exp_\nu^{-1}} & & \downarrow{\scriptstyle exp_\nu} \\ U \times [0,1] & \xrightarrow{\ F\ } & U \cong M - \mathrm{Cu}(N) \end{array}$$

The map F can be defined by taking the compositions

$$F = \exp_\nu \circ H \circ \exp_\nu^{-1}.$$

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$$\mathrm{Cu}\,(N/G) \cong \mathrm{Cu}(N)/G$$

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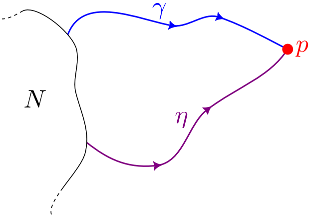

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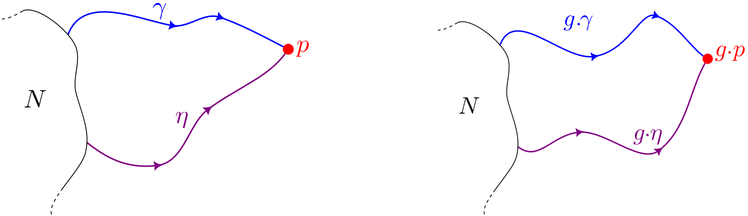

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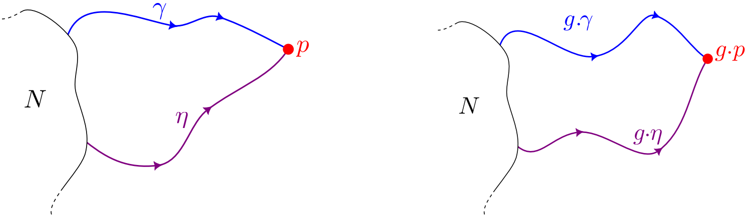

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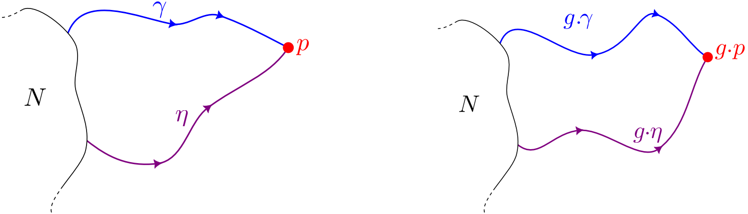

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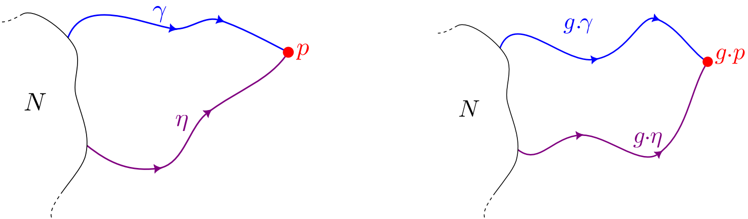

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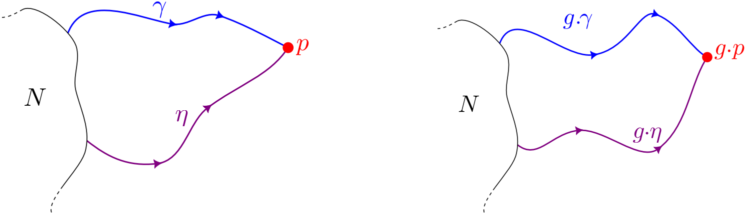

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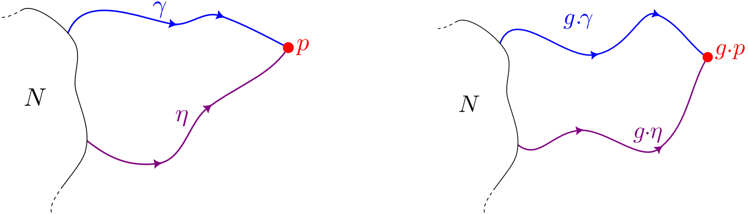

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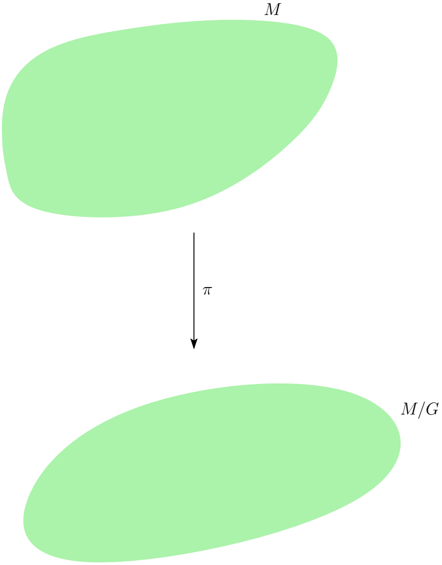

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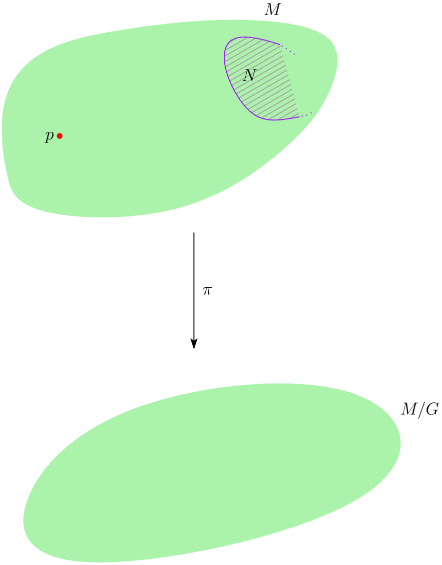

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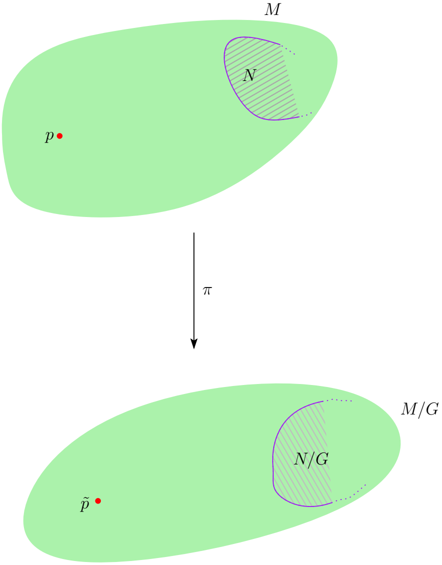

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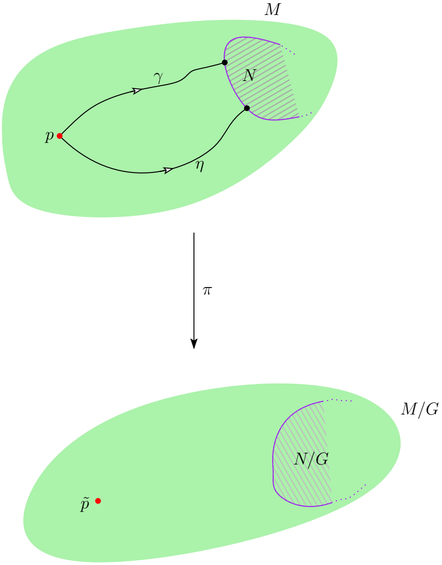

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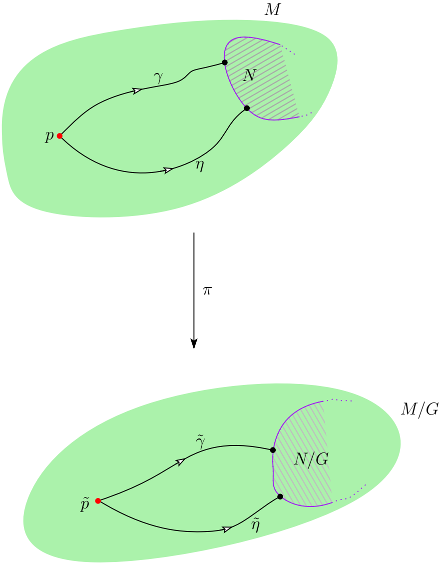

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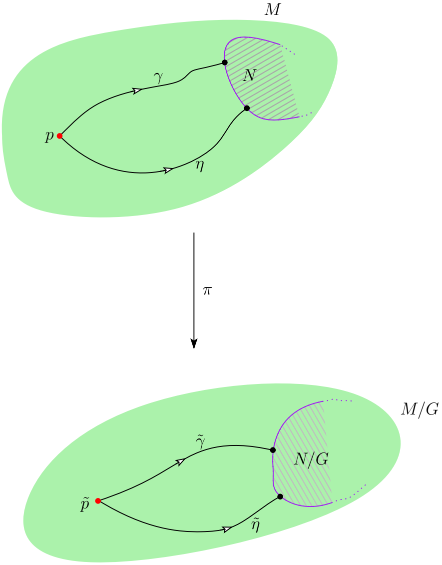

Problems in the approach

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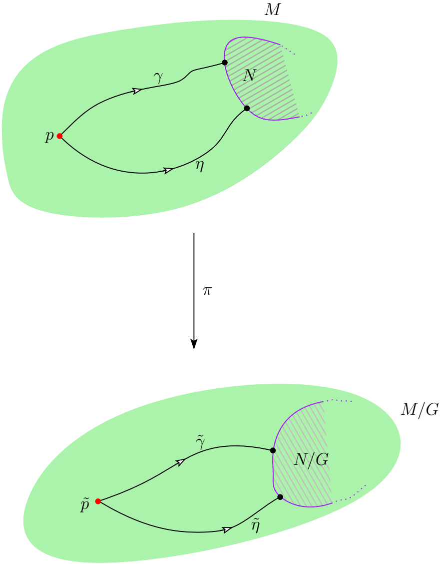

Problems in the approach

1. Why is $(\pi \circ \gamma)$ a distance minimal geodesic?

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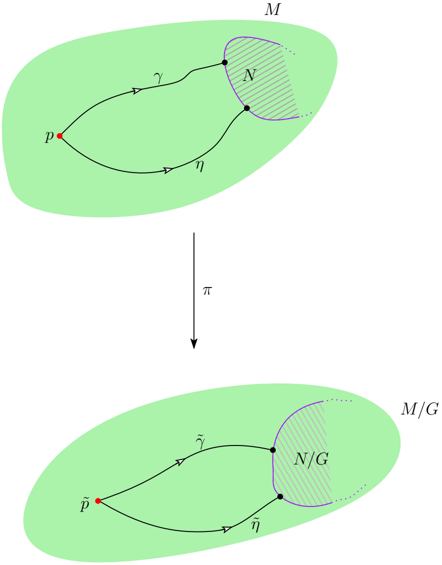

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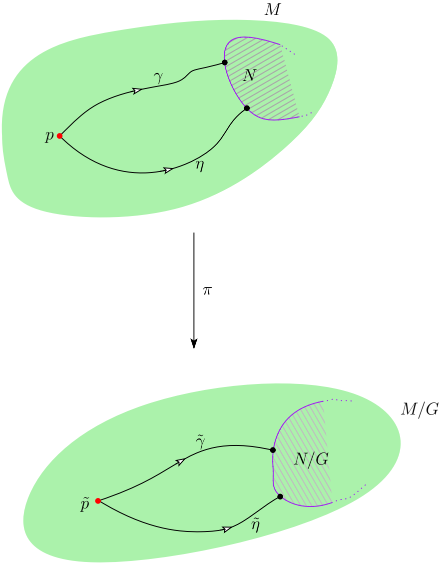

Problems in the approach

1. Why is $(\pi \circ \gamma)$ a distance minimal geodesic?
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3. The same for the lifts.

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- dg maps $\mathscr{H}_p$ to $\mathscr{H}_{g\cdot p}$.

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Proposition (Uniqueness of horizontal lift)

If $\gamma : [-1, 1] \to B$ is a smooth curve such that $\gamma(0) = b$ and $e_0 \in \pi^{-1}(b)$, then there is a unique horizontal lift $\tilde{\gamma}$ through $e_0 \in E$.

Geodesics on M and M/G

Geodesic on M and M/G

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Theorem

There is a one-to-one correspondence between the geodesics on M/G and geodesics on M which are horizontal.

Proof of the main theorem

$\mathrm{Se}(N)/G \subseteq \mathrm{Se}(N/G)$

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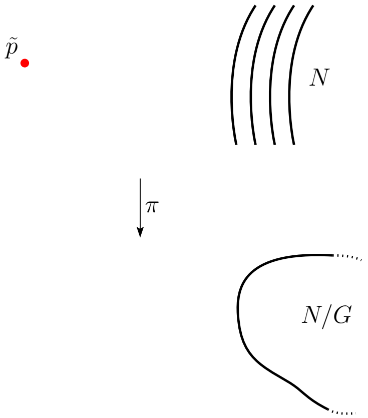

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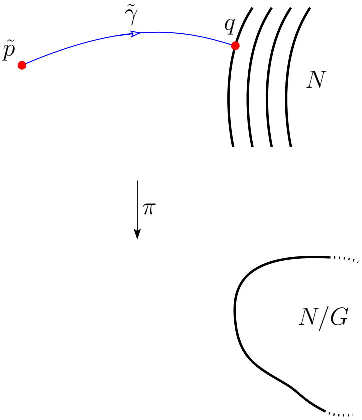

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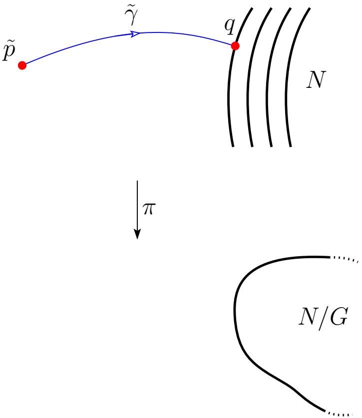
- $\tilde{\gamma}$ is a geodesic and $l(\tilde{\gamma}) = d(\tilde{p}, N)$



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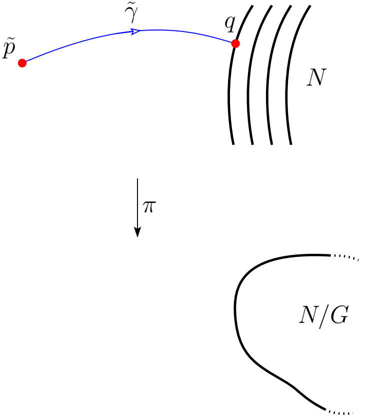

- $\tilde{\gamma}$ is a geodesic and $l(\tilde{\gamma}) = d(\tilde{p}, N)$ $\implies \tilde{\gamma}'(1) \in \mathscr{H}_{\tilde{\gamma}(1)}$

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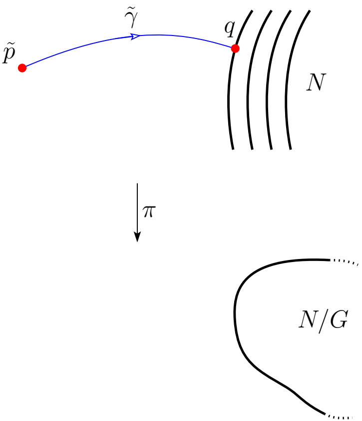

Lemma (O'Neil)

If $\tilde{\gamma}$ is a geodesic on E and $\tilde{\gamma}'(0) \in \mathscr{H}_{\tilde{\gamma}(0)}$, then for all t, $\tilde{\gamma}'(t) \in \mathscr{H}_{\tilde{\gamma}(t)}$ and $\pi \circ \tilde{\gamma}$ is a geodesic on B. Moreover, the length is preserved.

- $\tilde{\gamma}$ is a geodesic and $l(\tilde{\gamma}) = d(\tilde{p}, N)$ $\Longrightarrow$ $\tilde{\gamma}'(1) \in \mathscr{H}_{\tilde{\gamma}(1)}$

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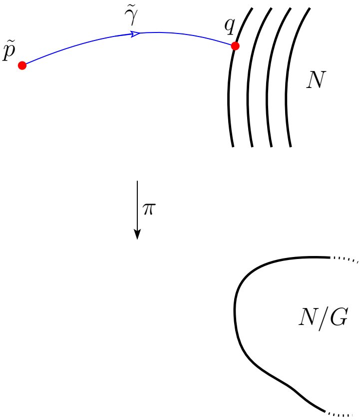

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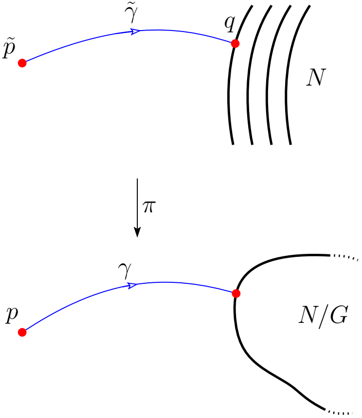

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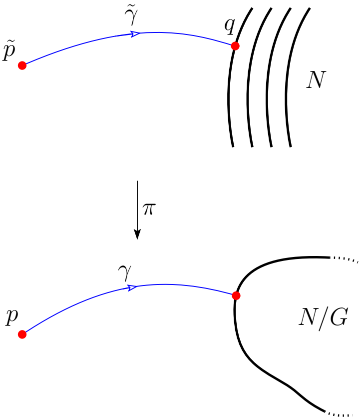

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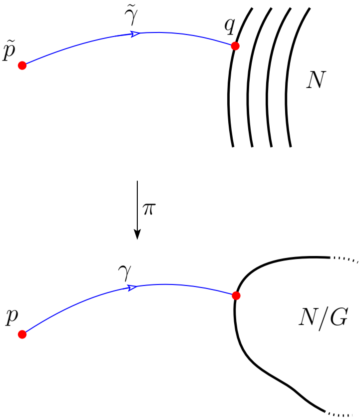

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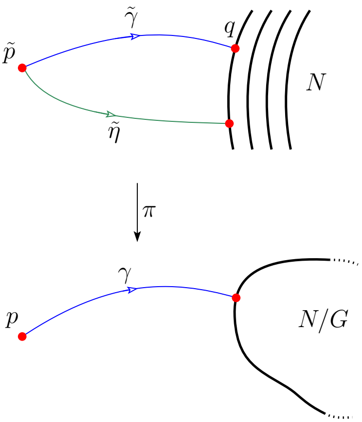
- If $\tilde{p} \in \mathrm{Se}(N)$,

$\mathrm{Se}(N)/G \subseteq \mathrm{Se}(N/G)$



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- If $\tilde{p} \in \mathrm{Se}(N)$, then there exists two N-geodesic, say $\tilde{\gamma}$ and $\tilde{\eta}$.

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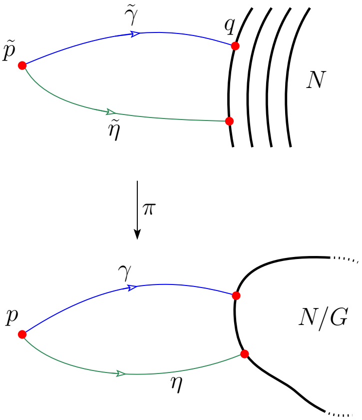

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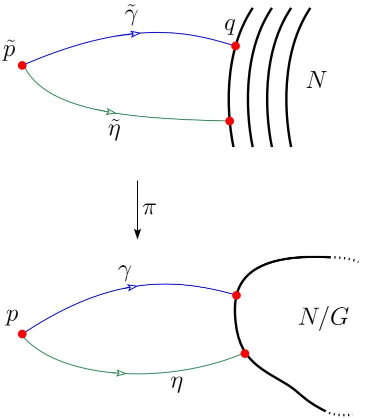

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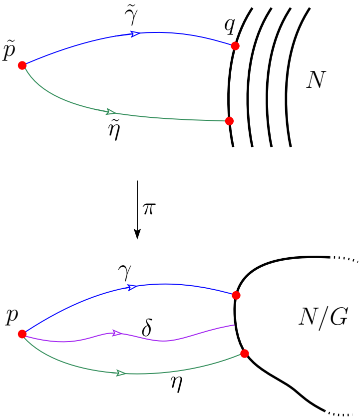

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- If $\tilde{p} \in \mathrm{Se}(N)$, then there exists two N-geodesic, say $\tilde{\gamma}$ and $\tilde{\eta}$. Due to uniqueness of horizontal lift, both will project to distinct geodesic and lengths are same. Note that $\tilde{\gamma}$ is an N-geodesic implies γ will be an N/G-geodesic. Otherwise, $\exists\ \delta$ which is N/G geodesic joining p to N/G

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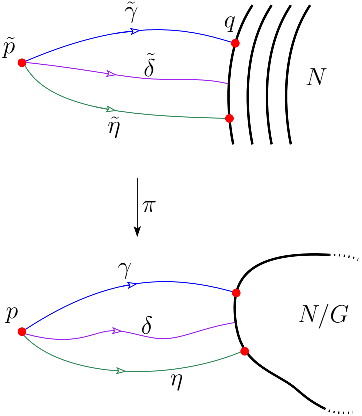

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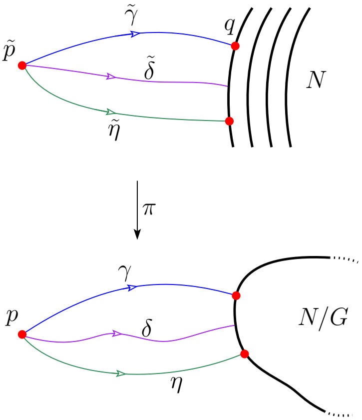

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Applications

Cut locus of projective planes

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Theorem (Basu, S. and Prasad, S.)

The cut locus of $\tilde{X}(d) \subseteq \mathbb{S}^{2n+1}$ *is* $\mathbb{Z}_d^{\star(n+1)} \times_{\mathbb{Z}_d} \mathbb{S}^1$.

Reference

S. Basu and S. Prasad, *A connection between cut locus, Thom space and Morse-Bott functions.* https://arxiv.org/abs/2011.02972 accepted in Algebraic & Geometric Topology.

R. S. Kulkarni and J. W. Wood, *Topology of nonsingular complex hypersurfaces*, Adv. in Math., 35 (1980), pp. 239–263.

Thank you for your attention!