Morse-Bott Flows and Cut Locus of Submanifolds

(based on joint work with Dr. Somnath Basu)

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Outline of the talk

1. Geometric aspects of the cut locus

2. Topological aspects of the cut locus

Geometric aspects of the cut locus

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The $\mathrm{Hess}_p(f)$ is non-degenerate in the direction normal to N at p means for any $V \in (T_pN)^\perp$ there exists $W \in (T_pN)^\perp$ such that $\mathrm{Hess}_p(f)(V, W) \neq 0$.

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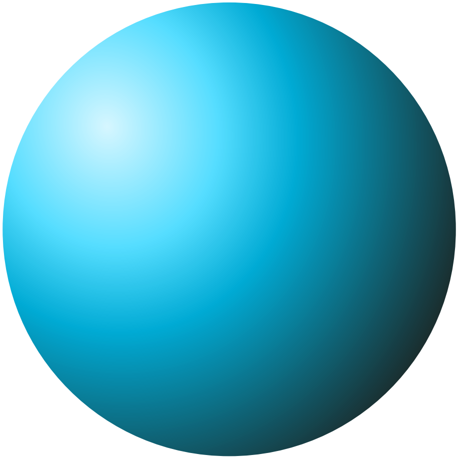

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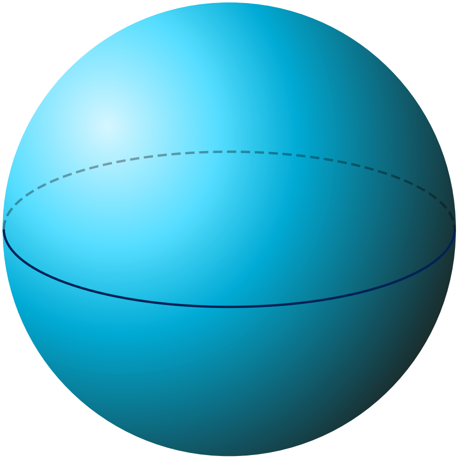

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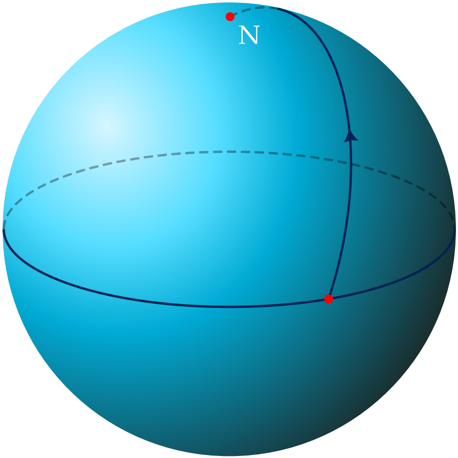

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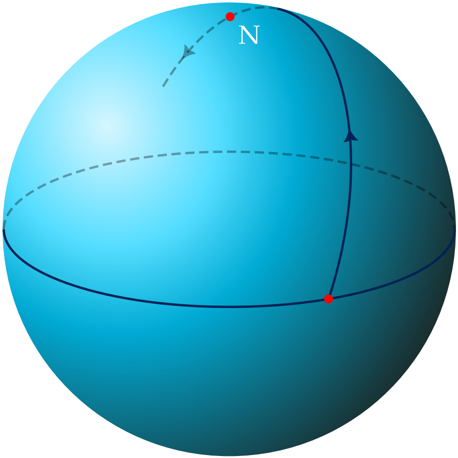

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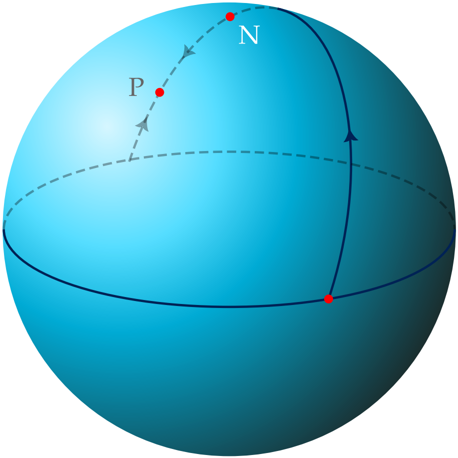

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Theorem (Basu S., Prasad S., 2021)

For a complete Riemannian manifold M and a compact submanifold N of M,

$$\overline{\mathrm{Se}(N)} = \mathrm{Cu}(N).$$

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Consider the distance squared function

$$f : M(n,\mathbb{R}) \to \mathbb{R}, \;\; A \mapsto d^2(A, O(n,\mathbb{R})).$$

- The function is $f(A) = n + \mathrm{tr}\left(A^T A\right) - 2\mathrm{tr}\left(\sqrt{A^T A}\right)$.
- It is differentiable at A if and only if A is invertible.
- It is a Morse-Bott function with critical submanifold as $O(n,\mathbb{R})$.
- If $\gamma(t)$ is an integral curve of $-\nabla f$ initialized at A, then

$$\frac{d\gamma}{dt} = -2\gamma(t) + 2\left(\gamma(t)^T\right)^{-1}\sqrt{\gamma(t)^T\gamma(t)}. \tag{1}$$

- The solution of (1) given by

$$\gamma(t) = Ae^{-2t} + (1-e^{-2t})A\left(\sqrt{A^T A}\right)^{-1}, \ \gamma(0) = A. \tag{2}$$

- The flow line $\gamma(t)$ deforms $GL(n,\mathbb{R})$ to $O(n,\mathbb{R})$.
- The separating set of $O(n,\mathbb{R})$ in $M(n,\mathbb{R})$ is set of singular matrices and as it is closed, the cut locus is the same.

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The cut locus $\mathrm{Cu}(N)$ is a strong deformation retract of $M - N$. In particular, $(M, \mathrm{Cu}(N))$ is a good pair and the number of path components of $\mathrm{Cu}(N)$ equals that of $M - N$.

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$$\mathbf{s} : S(\nu) \to [0, \infty], \ \mathbf{s}(v) := \sup\{t \in [0, \infty) \,|\, \gamma_v|_{[0,t]} \text{ is an } N\text{-geodesic}\},$$

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$$U_0(N) := \{av : 0 \le a < \mathbf{s}(v),\ v \in S(\nu)\}.$$

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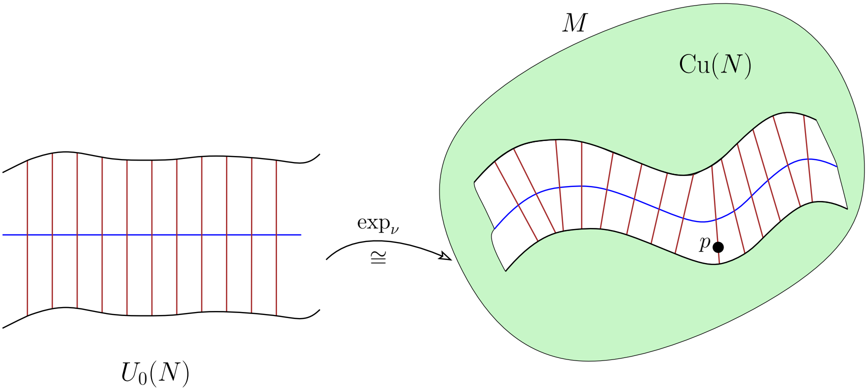
Note that $\exp_\nu$ is a diffeomorphism on $U_0(N)$ and set $U(N) = \exp_\nu(U_0(N)) = M - \mathrm{Cu}(N)$.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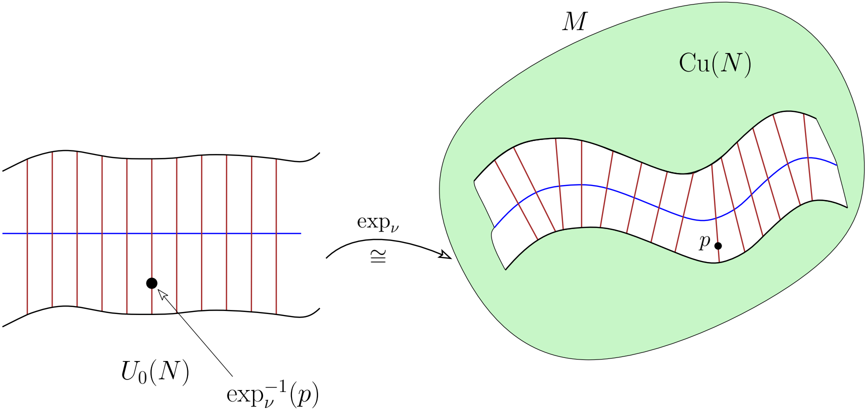

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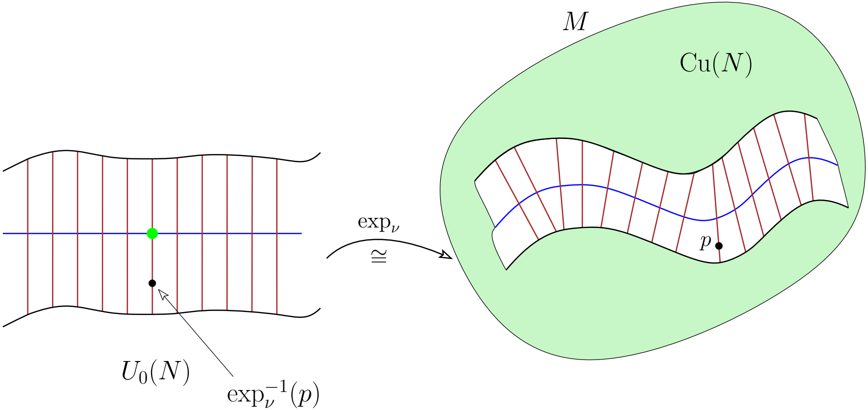

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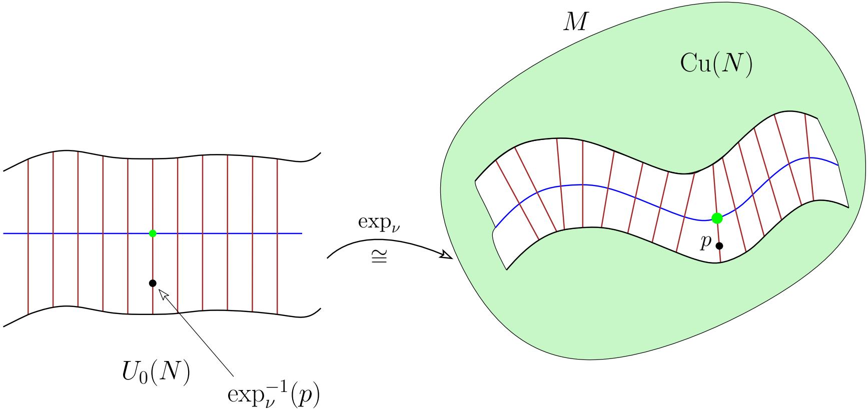
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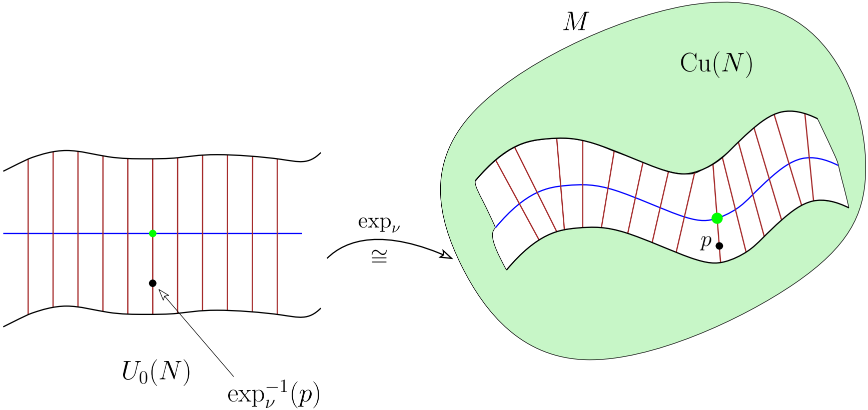

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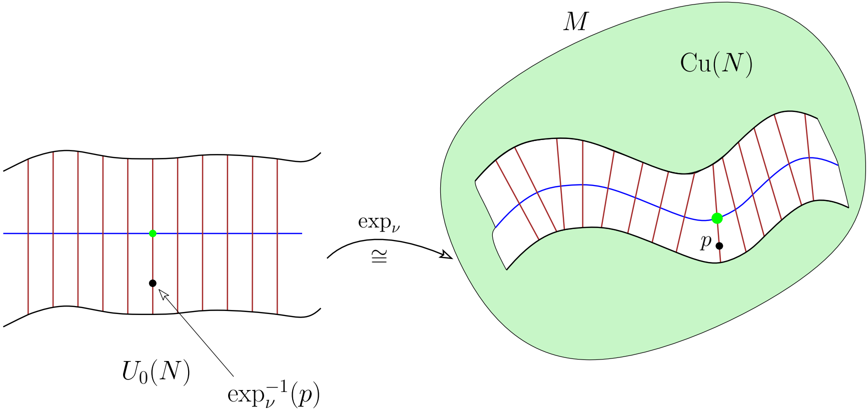

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$$H : U_0(N) \times [0,1] \to U_0(N), ((p, av), t) \mapsto (p, tav).$$

Now consider the following diagram:

$$\begin{array}{ccc} U_0(N) \times [0,1] & \xrightarrow{\quad H \quad} & U_0(N) \\ {\scriptstyle \exp_\nu^{-1}} \big\uparrow & & \big\downarrow {\scriptstyle exp_\nu} \\ U \times [0,1] & \xrightarrow{\quad F \quad} & U \cong M - \mathrm{Cu}(N) \end{array}$$

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The map F can be defined by taking the compositions

$$F = \exp_\nu \circ H \circ \exp_\nu^{-1}.$$

Topological aspects of the cut locus

Thom space

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Definition (Thom space)

Let $\pi : E \to B$ be a real vector bundle over a paracompact space B with a metric. Let $D(E)$ be the unit disk bundle and $S(E)$ be the unit sphere bundle.

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Remark

If B is compact, then $\mathrm{Th}(E)$ is the one point compactification of E.

Definition (Rescaled exponential)

The *rescaled exponential* map is defined to be

$$\widetilde{\exp} : D(\nu) \to M,\ (p, v) \mapsto \begin{cases} \exp_p(\mathbf{s}(\hat{v})v), & \text{if } v = \|v\|\hat{v} \\ p, & \text{if } v = 0. \end{cases}$$

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Since $\mathbf{s}$ is continuous, the rescaled exponential is also continuous and is surjective. Also note that $\widetilde{\exp}(S(\nu)) = \mathrm{Cu}(N)$.

The Main Theorem

Theorem (Basu S., Prasad S., 2021)

Let N be an embedded submanifold inside a closed, connected Riemannian manifold M. If ν denotes the normal bundle of N in M, then there is a homeomorphism

$$\widetilde{\exp} : D(\nu)/S(\nu) \xrightarrow{\cong} M/\mathrm{Cu}(N).$$

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3. Let N be a smooth homology k-sphere, $k > 0$, embedded in a smooth Riemannian manifold M homeomorphic to S^d. If $d \geq k+3$, then the cut locus $\mathrm{Cu}(N)$ is homotopy equivalent to S^{d-k-1}.

Thank You for your attention!