Large Mass Hierarchy from a Small Extra Dimension

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Abstract

The pivotal point of this report is to review the extra dimensional solution to the Hierarchy problem and associated Fine tuning problem in Higgs' mass.The report gives an short glimpse over the original Kaluza-Klein's idea of unifying gravity and electromagnetism using extra dimension.Therafter, the **ADD Model** with its pros and cons is reviewed briefly. Furthermore, the report also includes an elaborate description of **Randall-Sundrum** scenario (RS) with two 3-branes and how it generates **Large Mass Hierarchy from a Small Extra Dimension** via an exponential wrap factor,arising due to the nonfactorizable background metric (a slice of AdS_5 spacetime).Owing to this nonfactorizablity (unlike ADD), the constraints on ADD is not applicable to RS scenario.Last but not the least Goldberger-Wise mechanism for radius stabilisation has been elucidated.

1 Introduction

There are four fundamental forces in nature viz. Gravity, Strong force, Weak force, Electromagnetism. Among them Strong and Weak force are short ranged while the other two are infinte ranged owing to the fact the force carrier quanta of gravity and electromagnetism (graviton and photon respectively) are massless.Now the strength of each force is not the same.Compared to other forces gravity is weak. We can give numerous example to demonstrate the fact. Like a person is standing on the floor-the gravitational pull of whole big earth is counterbalalanced by normal reaction of the floor which is electromagnetic in nature.Now qualitatively saying weak does not mean anything in scientific language, hence we need to quantify the weakness of gravity. One way is turn our head to look back at the value of coupling constant. But there is a hitch. The gravitational coupling constant is $G_{4D} = M_{Pl}^{-2} = 10^{-38} GeV^{-2}$-a seemingly small number. Yet is it? Note, it is a dimensionful quantity and we can easily choose an unit so that the number becomes big!! So we are not really in a position to compare using a dimensionful quantity.Hence we need to get a dimensionless quantity out of it. One choice would be $G_{4D}m_p^2 = \frac{m_p^2}{M_{Pl}^2} = 10^{-38} << 1$, (where m_p is

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mass of a proton) in a similar manner if weak scale is denoted by $\nu \simeq 10^3 GeV$ then $G_{4D}\nu^2 = \frac{\nu^2}{M_{Pl}^2} = 10^{-35} << 1$.Thus gravity is weak compared to Strong, Weak and Electromagnetic when we are dealing with elementary particles of masses of the order of $1GeV$ or $1TeV$.But if mass of elementary particle becomes as high as $10^{19}GeV$, we have $G_{4D}m^2 \simeq 1$.

Now the well celebrated Standard Model of particle physics does not incorporate gravity beacuse we do not have a renormalizable quantized theory of gravity.But this does not pose a problem in low energy scale owing to the weakness of the gravity. We can safely ignore it as long as the particles are far less heavy than Planck Mass Scale. Albeit we can not trust our Standard Model when the energy scale is $10^{19}GeV$.This puts a standard cutoff on our model.We can at best trust Standard Model upto Planck Scale.Therefore we have a grand desert streching from weak scale to Planck scale essentially with no new physics and we do not know exactly why this hieararchy is there i.e we do not know why gravity is weak. We can not answer this question staying within Standard Model.This huge difference in mass scale and the concerned problem is known as Hieararchy Problem which manifests itself when we do loop correction to Higgs mass.This problem arises in quantum field theory because of the quadratically divergent corrections to the Higgs field mass, which require an incredible finetuning in order to get the expected mass of a few hundreds GeV.

Loop correction to Higgs mass yields

$$m^2 = m_0^2 + (10^{19}GeV)^2 \tag{1}$$

where m is the physical mass of the higgs,$m = 125GeV$ while m_0 is bare mass.Pluggin in the experimental value of m we obtain $|(\frac{m}{m_0})^2| \simeq 10^{-38}$. Hence the ratio has to be very precisely tuned to get the value we obtain experimentally. So we have a finetuning problem.

In fact,Hieararchy problem and Finetuning problem of higgs mass are two sides of the same coin. Note, the fine tuning is needed because we have a cutoff at $M_{Pl} = 10^{19}GeV$ which is very high compared to weak scale and this is same as saying gravity is weak compared weak or strong force at TeV energy scale.

Hence, Hierarchy problem is all about

- Why weak scale (ν) is so small compared to $M_{Pl} = 10^{19}GeV$.
- In other words why gravity is so weak.
- Is there any solution to fine tunning problem? Can we aviod such fine tuning of parameter like Higgs' mass.?

2 Quadratic Divergence associated with Scalar Field

In this section we will demonstrate the loop correction to mass of a scalar field leads to quadratic ultraviolet divergence.Note we will not deal with higgs directly, rather we will take an example of simple scalar field to elucidate the keypoint.

Let us the consider the following lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \frac{\lambda}{4!}\Phi^4 \tag{2}$$

In momentum space, at zero order the propagator is given by:

$$D(p) = \frac{\imath}{p^2 - m^2} \tag{3}$$

Now we consider the one loop correction to the propagator. Using the Feynman rule in momentum space, we obtain for the following diagram:

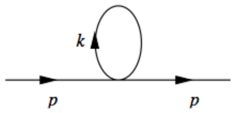

Figure 1: *One Loop Correction*

$$B = \frac{-\imath\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\imath}{k^2-m^2} \tag{4}$$

To evaluate the integral, we employ techniques from complex variable and power of contour integral. Let C_1 denote the real axis running from $-\infty$ to ∞ while C_2 denotes the same but along imagimary axis. Closing the contour at ∞ radius in first and third quadrant (They do not contain any poles!!) yields: As the contour does not

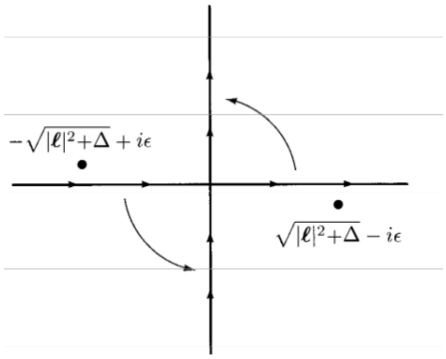

Figure 2: *Wick Rotation* ($l \equiv k$)

encloses any pole we have

$$\oint_{C_1-C_2} dz = 0 \tag{5}$$

But we have $\int_{C_1} dz = \int_{-\infty}^{\infty} dx$ and $\int_{C_2} dz = \imath\int_{-\infty}^{\infty} dy$. Hence we have,

$$\int_{-\infty}^{\infty} dx = \imath\int_{-\infty}^{\infty} dy \tag{6}$$

Using 6

$$\int_{-\infty}^{\infty} dk^0 = \imath_{-\infty}^{\infty} dk_M = \imath\int_{-\infty}^{\infty} dk_E^0 \tag{7}$$

where $z = k^0, x = k_M^0, y = k_E^0$.This is called **Wick Rotation**.Under this change of variable 4 becomes

$$B = \frac{-\imath\lambda}{2}\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} \tag{8}$$

The integral 8 is clearly divergent as the upper limit of integration goes to ∞. So we put a cut off at some arbitrary but large momentum Λ and evaluate the integral. Note, $d^4k = 2\pi^2 dk$. Using this we obtain

$$B = \frac{-\imath\lambda}{32\pi^2}\left(\Lambda^2 - m^2 ln(\frac{\Lambda^2+m^2}{m^2})\right) \tag{9}$$

Therefore B has a quadratic ultraviolet divergence. Now we can consider all the diagrams just made up of one loop placed repeatedly (See the following Feynman diagrams)

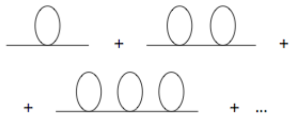

Figure 3: *Summing the Diagrams*

Now we can easily sum the whole series which actually gives the sum of corrections due to one loop structures only.Schematically, we have

$$D(p) + D(p)BD(p) + D(p)BD(p)BD(p) + ... = D(p)\sum_{n=0}^{\infty}[BD(p)]^n \tag{10}$$

Hence summing the R.H.S of 10

$$D(p)+D(p)BD(p)+D(p)BD(p)BD(p)+... = D(p)\frac{1}{1-BD(p)} = \frac{\imath}{p^2-m^2-\imath B} \tag{11}$$

Let $m_c^2 = \imath B = \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2 ln(\frac{\Lambda^2+m^2}{m^2})\right) \in \mathbf{R}$ using 9.

If we observe carefully 11 we see the pole of the modified propagator has been shifted from m to m' where m' is given by

$$m'^2 = m^2 + m_c^2 \tag{12}$$

Now 9 tell us m_c^2 goes as Λ^2 in leading order. Therefore we obtain

$$m'^2 = m^2 + \Lambda^2 \tag{13}$$

This equation 13 is what we quoted in introductory section 1 in connection with loop correction to Higgs mass. Note calculation for Higgs is fairly complex because of various coupling and associated gauge group but at the bottom, it is a scalar field. So we can kind of expect the same quadratic divergence in loop correction to Higgs mass too which is actually the case if one go through the calculation for Higgs field.

3 Kaluza-Klein Unification[1]

3.1 Kaluza's Idea

Before delving into the solution to Hierarchy problem, we introduce the notion of extra dimension from historical perspective.The first attempt to introduce extra dimensions is made by Kaluza in the year 1921 and a few years later by Klein (1926) to unify the two known forces of that era i.e gravity and electromagnetism.

To unify electricity and magnetism we kind of compactly write them in matrix form and they got promoteed to a second rank tensor which ideally shows the electric and magnetic field are not separate entities, they are components of one fundamental tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where $A^\mu = (V, \vec{A})$ is electromagnetic 4 potential.

$$\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

Kaluza had this idea on back of his mind. So he sort of try put together the metric tensor and A_μ of electromagnetism in a matrix form to achieve the unification. To accomodate A_μs he had to invoke one extra dimension.He wrote

$$g_{AB} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} & kA_0 \\ g_{10} & g_{11} & g_{12} & g_{13} & kA_1 \\ g_{20} & g_{21} & g_{22} & g_{23} & kA_2 \\ g_{30} & g_{31} & g_{32} & g_{33} & kA_3 \\ kA_0 & kA_1 & kA_2 & kA_3 & k \end{bmatrix}$$

Furthermore, he made the following assumptions:

- The model respects Einstein's vision that force is nothing but a manifestation of geometry. So we have A_μs as a part of 5 dimensional metric tensor.
- The mathematics of General Relativity is unaltered, merely extended to 5th dimension.
- 5 dimensional metric tensor does not depend upon 5th coordinate i.e the extra dimension.

The 4+1 D geodesic equation reads

$$\frac{d^2x^A}{ds^2} + \tilde{\Gamma}^A_{BD}\frac{dx^B}{ds}\frac{dx^D}{ds} = 0 \tag{14}$$

where the latin indices run from 0 to 4 and $\tilde{\Gamma}$ represents the 4+1 D affine connection. In the expression of $\tilde{\Gamma}$ we have indices running from 0 to 4. We can break any summation over repeated indices in two parts-one part containing the sum over indices running from 0 to 3, hence yielding standard objects of our 4D theory while other part is due to extra dimension. In this fashion we can extract 4 dimensional affine connection Γ from $\tilde{\Gamma}$. Using this we recast 14 in following form:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = kF^\lambda_\mu\frac{dx^\mu}{ds}\frac{dx^4}{ds} - kg^{\lambda 4}\partial_\nu A_\mu\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{15}$$

The first term of the right hand side of 15 looks like familiar Lorentz force if we interpret $k\frac{dx^4}{ds}$ as $\frac{e}{m}$. But the problem with this is $\frac{dx^4}{ds}$ is not a scalar quantity in 5D theory. Hold on...if we think in terms of 5D theory then A_μ is also not vector.They transform as a tensor in 5D theory. But there is a way out-we can alleviate the problem if we restrict our coordinate transformation in 3+1 D i.e we will consider coordinate transformation which mixes first 4 coordinates, leaving the extra dimension as it is. Under all such restricted coordinate transformation $\frac{dx^4}{ds}$ is indeed a scalar.Still there is a second term on right hand side of 15 hanging around which is nontensorial with respect to 4D theory. If the theory has to admit the law of general covariance we can not have such nontensorial term which leads to $g^{\lambda 4} = 0$.Now we will explicitly calculate the inverse metric and show $g^{\lambda 4} = 0$ leads to nonsense results.

To calculate the inverse metric we take the following ansatz:

$$g^{AB} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1/k \end{array}\right]$$

Now we have

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1/k \end{array}\right]\left[\begin{array}{c|c} g_{\mu\nu} & kA_\mu \\ \hline kA_\mu & k \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I}_4 & 0 \\ \hline 0 & 1 \end{array}\right]$$

Hence we must have $A^{\mu\nu} = g^{\mu\nu}$ i.e inverse of 4D metric.Plugging that in we have

$$\left[\begin{array}{c|c} g^{\mu\rho} & 0 \\ \hline 0 & 1/k \end{array}\right]\left[\begin{array}{c|c} g_{\mu\nu} & kA_\mu \\ \hline kA_\mu & k \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I}_4 & kA^\rho \\ \hline kA^\rho & 1 \end{array}\right]$$

Hence to have identity we must allow $A^\mu = 0$ which implies $A_\mu = 0$ which does not make sense!!So we have serious trouble if we want to unify in this fashion. The metric Kaluza wrote down needs modification.

3.2 Klein's Modification

The awaited modification came in the year 1926 in the hand of Klein. Klein did the following modification to identify

$$g^{AB} = \begin{bmatrix} g^{00} & g^{01} & g^{02} & g^{03} & -A^0 \\ g^{01} & g^{11} & g^{12} & g^{13} & -A^1 \\ g^{02} & g^{12} & g^{22} & g^{23} & -A^2 \\ g^{03} & g^{13} & g^{23} & g^{33} & -A^3 \\ -A^0 & -A^1 & -A^2 & -A^3 & 1/k + A_\mu A^\mu \end{bmatrix}$$

In a compact form we can write

$$g^{AB} = \left[\begin{array}{c|c} g^{\mu\nu} & -A^\mu \\ \hline -A^\nu & 1/k + A_\mu A^\mu \end{array}\right]$$

The inverse of g^{AB} is found to be

$$g_{AB} = \begin{bmatrix} g_{00} + kA_0A_0 & g_{01} + kA_0A_1 & g_{02} + kA_0A_2 & g_{03} + kA_0A_3 & kA_0 \\ g_{01} + kA_0A_1 & g_{11} + kA_1A_1 & g_{12} + kA_1A_2 & g_{13} + kA_1A_3 & kA_1 \\ g_{02} + kA_0A_2 & g_{12} + kA_1A_2 & g_{22} + kA_2A_2 & g_{23} + kA_2A_3 & kA_2 \\ g_{03} + kA_0A_3 & g_{13} + kA_1A_3 & g_{23} + kA_2A_3 & g_{33} + kA_3A_3 & kA_3 \\ kA_0 & kA_1 & kA_2 & kA_3 & k \end{bmatrix}$$

which can be recast again in a compact form via

$$g_{AB} = \left[\begin{array}{c|c} g_{\mu\nu} + kA_\mu A_\nu & kA_\mu \\ \hline kA_\nu & k \end{array}\right]$$

Note, writing in the compact form facilitates calculation by hand. Like, we can easily show g^{AB} is the inverse of g_{AB}:

$$\left[\begin{array}{c|c} g^{\mu\nu} & -A^\mu \\ \hline -A^\nu & 1/k + A_\mu A^\mu \end{array}\right] \left[\begin{array}{c|c} g_{\nu\rho} + kA_\nu A_\rho & kA_\nu \\ \hline kA_\rho & k \end{array}\right] =$$

$$\left[\begin{array}{c|c} \delta^\nu_\rho + kA^\mu A_\rho - A^\mu kA_\rho & kA^\mu - kA^\mu \\ \hline -A^\rho + kA^\nu A_\nu A_\rho + A_\rho + kA^\mu A_\mu A_\rho & -kA^\nu A_\nu + 1 + kA_\mu A^\mu \end{array}\right] = \mathbf{I}$$

The determinant can easily be computed as

$$det(g_{AB}) = kdet(g_{\mu\nu}) \tag{16}$$

The geodesic equation is found out be

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = kF^\lambda_\mu \frac{dx^\mu}{ds}\frac{dx^4}{ds} + kA_\mu F^\lambda_\nu \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{17}$$

Note we do not have any nontensorial term as opposed to Kaluza's metric. This equation is tensorial in nature althouth the second term on right hand side does not have any 4D analogue.

3.3 Gauge Transformation

We have considered restricted coordinate transformation.So the natural question is to ask how does an infintesimal change in 5th coordinate manifest itself in 4D theory. Consider an infinitesimal change in 5th coordinate

$$x'^4 = x^4 + \zeta(x^\mu) \tag{18}$$

$$\delta dx^4 = \partial_\mu \zeta dx^\mu \tag{19}$$

The line element is given by

$$ds^2 = g_{AB}dx^A dx^B = g_{\mu\nu}dx^\mu dx^\nu + kA_\mu A_\nu dx^\mu dx^\nu + 2kA_\nu dx^\nu dx^4 + kdx^4 dx^4 \quad (20)$$

Under the infinitesimal transformation ds^2 should remain invariant. On the other hand the 4D subspace line element given by $g_{\mu\nu}dx^\mu dx^\nu$ remains invariant which forces A_μ to transform in a specific manner:

$$A'_\mu = A_\mu - \partial_\mu \zeta(x^\alpha) \quad (21)$$

This 21 is precisely the gauge transformation of A_μ, electromagnetic 4-potential.

3.4 Kaluza-klein Action

We can recast the whole idea by writing down the action for 5D theory.In electromagnetism we do not need separate actions for Electricity and Magnetism, rather we have a single action from which we can derive field equations (Maxwell's equation) governing electricity and magnetism simulataneously. Writing down an single action brings out the underlying unification between Electricity and Magnetism. We can not talk about them invidually, rather we need to see them as different manifestion of same physical quantity which is in this case $F_{\mu\nu}$. So motivated by this we try to write down a single action which would describe gravity and electromagnetism in one go.

Now the standard 4D general relativity action can be written as

$$S_{GR} = \frac{1}{16\pi G_{4D}} \int d^4x\sqrt{-g}R \quad (22)$$

Kaluza-Klein action is a simple genralisation of 22 to 5D given by

$$S_{KK} = \frac{1}{16\pi G_{5D}} \int d^5x\sqrt{-\tilde{g}}\tilde{R} \quad (23)$$

Now the 5D Ricci scalar can neatly broken down and can be written in terms of 4D Ricci scalar and $F_{\mu\nu}$s:

$$\tilde{R} = R - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad (24)$$

Moreover,we have $\tilde{g} = kg$ where g is the determinant of the 4D metric and $\tilde{g}$ is determinant of 5D metric.

Hence, we can plug 24 and the determinant in 23 which allows us to integrate over the extra dimension. Klein assumed the extra dimension to be compactified on a circle of radius R.

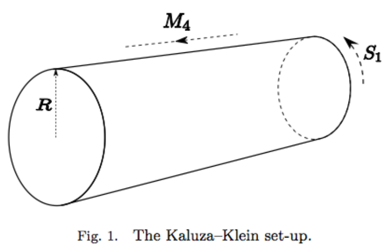

Figure 4: *Compactified Extra Dimension*

With this assumption, the action 23 can be written as

$$S_{KK} = \frac{\sqrt{k}}{16\pi G_{5D}} \int dx^4 \int d^4x\sqrt{-g}\left[R - \frac{k}{4}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{25}$$

Integrating over the extra dimension we obtain an effective action given by

$$S_{eff} = \frac{2\pi R\sqrt{k}}{16\pi G_{5D}} \int d^4x\sqrt{-g}\left[R - \frac{k}{4}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{26}$$

Comparing 26 with the original GR action 22 we obtain a very important result which is later exploited in ADD model:

$$G_{4D} = \frac{G_{5D}}{2\pi R\sqrt{k}} \tag{27}$$

Note, $2\pi R\sqrt{k}$ is a measure of compactified volume of extra dimension. So we can write 27 in following manner which is really suggestive as we will see later in ADD model:

$$G_{4D} = \frac{G_{5D}}{V_1} \tag{28}$$

4 ADD Model[2]

Now we will be back with Hierarchy problem and we will show how one can use the idea of extra dimension to solve one of central problems in Particle Physics.

4.1 The Model-Setting up the Stage

In the year 1998, N.Arkani-Hamed, S.Dimopoulos, G.Dvali **(ADD)** proposed a higher dimensional model to solve hierarchy problem.They assumed the space-time geometry with 4+n dimension is factorizable.The factorizablity implies the metric can be neatly decoupled into two parts; one for the usual 4 dimensional space-time while the other part accounts for extra n spatial dimensions. Note in Kaluza-Klein theory we have one extra dimension compactified on a circle. Hence it is a more general set up that Kaluza or Klein had originally imagined.The line element is given by:

$$ds^2 = \tilde{g}_{AB}dx^A dx^B = g_{\mu\nu}dx^\mu dx^\nu + \gamma_{ij}dy^i dy^j \tag{29}$$

The factorizablity leads to several simplifications:

- The Ricci scalar for the 4+n dimensional theory can be written down as

$$R[\tilde{g}_{AB}] = R[g_{\mu\nu}] + R[\gamma_{ij}] \tag{30}$$

- The determinant of the metric is given by

$$\tilde{g} = g\gamma \tag{31}$$

  where g represents determinant of $g_{\mu\nu}$ and γ represents determinant of γ_{ij}.

Now we write down action for 4+n dimesional theory:

$$S = \frac{1}{16\pi G_{4+n}} \int d^{4+n}x\sqrt{-\tilde{g}}R[\tilde{g}_{AB}] \tag{32}$$

where G_{4+n} is the fundamental higher dimensional gravitational constant. As we did in Kaluza Klein theory we assume the extra dimensions are compactified in a finite volume i.e

$$V_n = \int d^n y\sqrt{\gamma} \tag{33}$$

Note, the idea and mathematical structure is very alike the idea proposed by Kaluza-Klein. Here also, because the determinant of the metric as well as the Ricci scalar can be written down as having two parts,one being the function of extra dimensions only, we can easily integrate over extra dimensions and using 33 come up with an effective 4D action

$$S_{eff} = \frac{V_n}{16\pi G_{4+n}} \int d^4x\sqrt{-g}\left(R[g_{\mu\nu}] + \frac{1}{V_n}\int d^ny\sqrt{\gamma}R[\gamma_{ij}]\right) \tag{34}$$

which can be compared with

$$S_{GR4D} = \frac{1}{16\pi G_{4D}} \int d^4x\sqrt{-g}R[g_{\mu\nu}] \tag{35}$$

. Comparing 34 and 35 we relate the fundamental gravitational coupling constant to the 4D one by

$$\frac{V_n}{G_{4+n}} = \frac{1}{G_4} \tag{36}$$

This indeed generalises the set up with one extra dimension where we have 28.

4.2 Solution to Hierarchy Problem in ADD

A little bit of dimensional analysis using the fact $[R] = m^2$ ($\hbar = c = 1$) reveals

$$V_n = \left(\frac{1}{\mu_c}\right)^n \tag{37}$$

$$G_{4D} = \frac{1}{M_{Pl}^2} \tag{38}$$

$$G_{4+n} = \frac{1}{M^{n+2}} \tag{39}$$

where M_{Pl} is effective 4 dimensional (reduced) Planck scale, $M_{Pl} = 2\times10^{18}GeV$, M is the fundamental 4+n dimensional Planck scale and $\frac{1}{\mu_c}$ represents the compactification length scale.

We recast 36 into a familiar form as quoted in [4] using mass scales

$$M_{Pl}^2 = \frac{M^{n+2}}{\mu_c^n} \tag{40}$$

As we mentioned earlier there is a problem concerning loop correction of higgs mass because M_{Pl} is so high compared to 1 TeV.We can believe on our standard model at best upto M_{Pl} energy and if we do that we encounter the hierarchy problem in 4 dimension.If the cut off scale is of the oder of TeV, the problem would not exist because in that case Standard model is valid upto TeV scale and as a result loop correction can not render the higgs mass as high as it would do otherwise.So would it have been like the cut off is at 1TeV, we do not need to finetune the parameter like higgs bare mass.

Keeping these in mind, ADD Model sets the fundamental Planck scale $M = 1TeV$ which solves the hierarchy issue in fundamental theory.Now we have μ_c,the compactification scale in our hand,we can set μ_c to such a value so that 40 leads to $M_{Pl} = 10^{19}GeV$. A simple calculation reveals

$$\mu_c = 10^{-\frac{32}{n}}TeV \tag{41}$$

The length scale of compactification is given by

$$r_c = \frac{1}{\mu_c} = 10^{\frac{32}{n}}10^{-17}cm \tag{42}$$

Hence we have solution to the fine tuning problem of higgs bare mass.In this model gravity is not weak in fundamental higher dimensional theory. But effectively in 4D, gravity becomes diluted because of extra dimensional bulk voume.The volume factor reduces M to M_{Pl} in 4D.

Now question how we can detect the extra dimension.The model constrains all the force quantas except gravity to reside on the 3-brane.[2]So only gravity can propagate through bulk.So we need to probe gravity to ascertain the existence of extra dimensions.

4.3 Experimental Constraint on Number of Extra Dimensions in ADD

If we have 4+n dimensions with factorizable geometry, the gravitational law reads

$$F \propto \frac{1}{r^{2+n}} \tag{43}$$

We can show the force will behave like this only when the distance between two objects are comparable to compactification length scale. Now we have checked the law of gravity, the usual $\frac{1}{r^2}$ law only upto 1mm.[3] So the compactification length scale has to be less than 1mm-this is the experimental bound on any factorizable model.

We go back to 42.Putting n=1 yields 10^{15} cm which is too large.Next we put n=2 to get $r_c = 1mm$. Hence the experimental least bound on number of extra spatial dimensions is 2.

4.4 Gravitational Field of a Point Mass in compactified 5D World[3]

This section is only meant for demonstration of certain things when we have factorizable geometry. Thus we only consider the presence of one extra dimension only. Suppose, a point mass M is located at (0,0,0,0). Without compactification, we will try to find the potential using Gauss Law. Because we have radial symmetry we can write

$$\oint_{S-3D} \vec{I}.\vec{ds} = I.R^3 vol(S^3(1)) = 2\pi^2 I R^3 \tag{44}$$

where S^3 denotes the volume of 3 dimensional surface which bounds a 4D ball of unit radius and $\vec{I}$ is the garvitational field. Now Gauss law reads

$$\oint_{S-3D} \vec{I}.\vec{ds} = -4\pi G_{5D} M \tag{45}$$

Using 44 in 45, we obtain $\vec{I} = \frac{-2G_{5D}M}{\pi R^3}\hat{r}$. Hence the potential is found out to be

$$V_g^5(R = \sqrt{x^2+y^2+z^2+w^2}) = \frac{-G_{5D}M}{R^2} \tag{46}$$

Now we compactify the extra dimension in a circle of radius a.Physically the situation is same as putting identical masses M at an interval of $2\pi a$ of the extra dimension i.e we have a series of mass M located at $(0,0,0,2n\pi a)$ where $n \in \mathbf{Z}$.

Hence,the potential at $(x, y, z, 0)$ can be written as

$$V_g^5(R) = \sum_{n=-\infty}^{n=\infty} \frac{-G_{5D}M}{x^2+y^2+z^2+4n^2\pi^2a^2} = \frac{-G_{5D}M}{2aR}\coth\left(\frac{R}{2a}\right) \tag{47}$$

[2]a p-brane is p dimensional hypersurface of any n dimensional bulk volume where $n \geq p$.So 3-brane refers to our 3+1 dimensional universe.

where $R^2 = x^2 + y^2 + z^2$.

Now we note for $R >> a$ we have $V_g^5(R) = \frac{-G_{5D}M}{2aR}$. Invoking 36 we have

$$V_g^5(R >> a) \propto \frac{-G_{4D}M}{R} \tag{48}$$

We recover our standard gravitational law. But if we probe into $R \simeq a$ or $R < a$ regime, we expand *coth* in terms of $\frac{R}{2a}$, we have $\coth\left(\frac{R}{2a}\right) = \frac{2a}{R} + \frac{R}{6a} + ...$Hence,

$$V_g^5(R < a) = \frac{-G_{5D}M}{R^2} - \frac{G_{5D}M}{12a^2} + ... \tag{49}$$

Now in 49, the first term is proportional to $\frac{-G_{4D}}{R^2}$. So only in the length scale comparable to compactification scale we can get a signature of potential of $\frac{1}{R^2}$ form i.e force becomes $\propto \frac{1}{R^3}$. So the presence of extra dimension can only only be probed if we get close enough to the scale of compactification.We have to probe gravity in that small length scale to ascertain the existence of extra dimension.Today all the experiments confirm that we have $\frac{1}{R^2}$ force law which means even if we have extra dimensions, we have not got into the length scale of compactification and thereby it puts a constraint on compactification scale, hence on no of minimum extra dimensions required in factorizable geometry.

4.5 Hierarchy in Disguise

It seems now ADD has achieved its aim, it solved the hierarchy problem. But there is a catch. We have a new parameter in our hand i.e μ_c and we have to set it to some value to get M_{Pl} from fundamental Planck scale M. As we will show now there is fine tuning associated with the value of this parameter.So we have solved the hierarchy between effective 4D Planck scale and electroweak scale (higgs mass) but we have to pay a price by introducing another geometric hierarchy i.e between μ_c, the compactification scale and electroweak scale.[4]

$$\frac{\mu_c}{\nu} = 10^{-\frac{32}{n}} \tag{50}$$

where ν is electroweak scale.Unless n is a huge number which is unlikely, we have indeed a fine tunning problem (see the ratio 50) -ADD model does not give answer to the question why $\frac{\mu_c}{\nu}$ takes so precise value to satisfy 40.

5 Randall Sundrum Model[4]

5.1 Prelude

ADD model being faced with such burning issues, people started to seek alternatives. The next novel idea came from Lisa Randall and Raman Sundrum in the year 1999. They proposed a new higher dimensional mechanism using nonfactorizable geometry. The model is distinct from ADD in following points:

1. As the geometry is nonfactorizable, the bound obtained on number of extra dimensions does not apply here. We have n=1 as opposed to $n \geq 2$.
2. There is no disguised hierarchy as we will see here hierarchy is generated via exponential function and we know, a very small value when raised to the power of e, it can yield a large value.So a small value of the new parameter of the theory suffices.We will come back to this issue after we explain the model.

Like ADD model, here also only gravity can propagate through bulk volume while other forces are confined to 3-brane.(Note, notations are slightly changed compared to what we have used in section 2.I have tried my best to keep you up with the notations with possible clarification where needed).

5.2 Set Up

- The geometry is nonfactorizable with following form of metric tensor:

$$ds^2 = G_{MN}dx^M dx^N = e^{-2\sigma(\phi)}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{51}$$

  where the greek index runs from 0 to 3 while the latin index runs from 0 to 4.
- The topology of the extradimension is $S^1/\mathcal{Z}_2$. The 5th coordinate is angular in nature with (x, ϕ) identified with $(x, -\phi)$.
- The 3-branes extending in x^μ direction are located at $\phi = 0$ and $\phi = \pi$.
- The metric induced on the branes is given by

$$g_{\mu\nu}^{vis} = G_{\mu\nu}(x^\mu, \phi = \pi) \tag{52}$$

$$g_{\mu\nu}^{hid} = G_{\mu\nu}(x^\mu, \phi = 0) \tag{53}$$

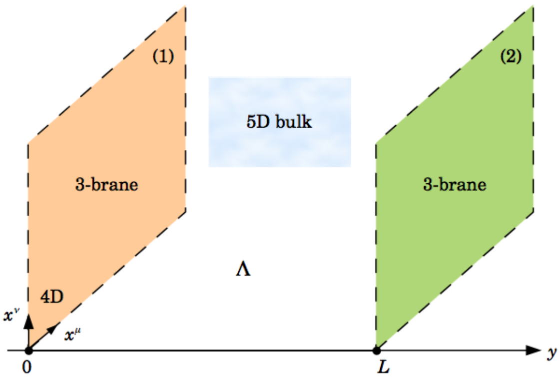

Figure 5: *Randall Sundrum Model[5]*

[3]

The classical action is given by

$$S = S_{gravity} + S_{vis} + S_{hid} \tag{54}$$

where

$$S_{gravity} = \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{-G}[-\Lambda + 2M^3 R] \tag{55}$$

$$S_{vis} = \int d^4x \sqrt{-g_{vis}}[\mathcal{L}_{vis} - V_{vis}] \tag{56}$$

$$S_{hid} = \int d^4x \sqrt{-g_{hid}}[\mathcal{L}_{hid} - V_{hid}] \tag{57}$$

Note, V_{vis} and V_{hid} is vacuum energy of the branes which act as a gravitational source even in absence of matter on brane.Without putting matter on the branes, we obtain the following equation by varying the action with respect to bulk metric:

$$\sqrt{-G}\left(R_{MN} - \frac{1}{2}RG_{MN}\right) = \frac{-1}{4M^3}\left[\Lambda\sqrt{-G}G_{MN} + V_{vis}\sqrt{-g_{vis}}g_{\mu\nu}^{vis}\delta_M^\mu\delta_N^\nu\delta(\phi - \pi)\right]$$

[3]Note in the figure $y = r_c\phi$

$$+\frac{-1}{4M^3}\left[V_{hid}\sqrt{-g_{hid}}g^{hid}_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N}\delta(\phi)\right] \tag{58}$$

We assume there exists a solution satisfying 4 dimensional Poincare invariance in x^μ direction.The ansatz satisfying this is given by 51. With this ansatz,we find

$$R_{44} - \frac{1}{2}RG_{44} = \frac{6\sigma'^2}{r_c^2} \tag{59}$$

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = \frac{\left(6\sigma'^2 - 3\sigma''\right)}{r_c^2}e^{-2\sigma(\phi)}\eta_{\mu\nu} \tag{60}$$

59 reduces to

$$\frac{6\sigma'^2}{r_c^2} = \frac{-\Lambda}{4M^3} \tag{61}$$

while 61 and 60 yield

$$\frac{3\sigma''}{r_c^2} = \frac{V_{hid}}{4M^3 r_c}\delta(\phi) + \frac{V_{vis}}{4M^3 r_c}\delta(\phi-\pi) \tag{62}$$

The solution to 61 consistent with orbifold symmetry is

$$\sigma = r_c|\phi|\sqrt{\frac{-\Lambda}{24M^3}} \tag{63}$$

Hence Λ turns out to be negative i.e bulk spacetime in between the branes is a slice of an AdS_5 geometry.

Keeping in mind that the coordinate ϕ is periodic, we find

$$\sigma'' = 2r_c\sqrt{\frac{-\Lambda}{24M^3}}\left[\delta(\phi) - \delta(\phi-\pi)\right] \tag{64}$$

where we have used following identities[5]:

$$\frac{d}{dy}(|y|) = sign(y) = \theta(y) - \theta(-y) \tag{65}$$

$$\frac{d^2}{dy^2}(|y|) = 2\delta(y) \tag{66}$$

Now we compare 62 and 64. To have a consistent solution we must have

$$V_{hid} = -V_{vis} = 24M^3 k \tag{67}$$

where k is a parameter determined by Λ via

$$k = \sqrt{\frac{-\Lambda}{24M^3}} \tag{68}$$

Our solution for the bulk metric is then

$$ds^2 = e^{-2kr_c|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{69}$$

5.3 Physical Implication

In this section we will derive the parameters of effective 4D theory in terms fundamental parameter M, k, r_c.We will see how the 5D Ricci scalar contains the 4D one, then we will integrate over extra dimension as done in ADD model to obtain the effective 4D action. Note $R[\eta_{\mu\nu}] = 0$, hence we need to perturb the Minkowski metric slightly to have

$$ds^2 = e^{-2kr_c|\phi|}\bar{g}_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{70}$$

where $\bar{g}_{\mu\nu} = \left(\eta_{\mu\nu} + \bar{h}_{\mu\nu}\right)$, $\bar{h}_{\mu\nu}$ denoting the tensor fluctuation over Minkowski spacetime.

We introduce a new variable $k|z| = e^{kr_c|\phi|} - 1$ [5] to obtain a conformal form of the metric:

$$ds^2 = g_{MN}dx^M dx^N = \frac{1}{(k|z|+1)^2}\left(\bar{g}_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{71}$$

So we have $g_{MN} = e^{-2A}\bar{g}_{MN}$ where $\bar{g}_{44} = 1$ and $A = ln(k|z|+1)$

Now for conformally related metric we have a nice formula[5]

$$E_{MN} = R_{MN} - \frac{1}{2}Rg_{MN} = \bar{R}_{MN} - \frac{1}{2}\bar{R}\bar{g}_{MN} + 3(\partial_M A\partial_N A + \partial_M\partial_N A)$$

$$-\bar{g}_{MN}(\bar{g}^{FR}\partial_F\partial_R A - \bar{g}^{FR}\partial_F A\partial_R A) \tag{72}$$

where $\bar{R}_{MN}$ is due to $\bar{g}_{MN}$

Hence,

$$R = -\frac{1}{2}g^{MN}E_{MN} = -\frac{1}{2}g^{MN}[\bar{E}_{MN} + ...] = -\frac{1}{2}(k|z|+1)^2\bar{g}^{MN}[\bar{E}_{MN} + ...] \tag{73}$$

Now we break the sum into two parts:one owing to μ, ν running for 0 to 3 while $M = N = 4$ is kept aside. Doing this we obtain

$$R = -\frac{1}{2}[(k|z|+1)^2\bar{g}^{\mu\nu}\bar{E}_{\mu\nu} - \frac{1}{2}(k|z|+1)^2\bar{g}^{44}\bar{E}_{44} + ... \tag{74}$$

Finally we have the desired result which shows how 5D Ricci scalar containd the 4D one.

$$R = (k|z|+1)^2\bar{R} + ... \tag{75}$$

Hence from 55 bulk action (without Λ) is given by

$$S_{gravity} \supset \int d^4x \int dz \frac{1}{(k|z|+1)^5}\sqrt{-\bar{g}}2M^3(k|z|+1)^2\bar{R} \tag{76}$$

We again go back to ϕ coordinate to obtain

$$S_{gravity} \supset \int d^4x\sqrt{-\bar{g}}\int_{-\pi}^{\pi} d\phi r_c e^{-2kr_c|\phi|}\bar{R} \tag{77}$$

Integrating ove ϕ

$$S_{eff} = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right)\int d^4x\sqrt{-\bar{g}}\bar{R} \tag{78}$$

Comparing it to the standard 4D GR action 35 we obtain three important results:

1. In effective Einstein action $\bar{g}_{\mu\nu}$ behaves as an effective metric.
2. We can relate reduced 4D planck scale to fundamental one via

$$M_{Pl}^2 = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right) \tag{79}$$

3. M_{Pl} depends weakly on r_c in the large kr_c limit.

5.4 Solution to Hierarchy

From 52 and 53 we obtain

$$g_{\mu\nu}^{vis} = e^{-2kr_c\pi}\bar{g}_{\mu\nu} \quad (80)$$

$$g_{\mu\nu}^{hid} = \bar{g}_{\mu\nu} \quad (81)$$

Now we will show the exponential factor sitting in the $g_{\mu\nu}^{vis}$ plays a crucial role in determining physical masses.

Here we consider a fundamental Higgs field governed by

$$S_{vis} \supset \int d^4x\sqrt{-g_{vis}}[g_{vis}^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - \nu_0^2)^2] \quad (82)$$

Using expression for $g_{\mu\nu}^{vis}$ from 80 we obtain

$$S_{vis} \supset \int d^4x\sqrt{-\bar{g}}e^{-4kr_c\pi}[\bar{g}^{\mu\nu}e^{2kr_c\pi}D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - \nu_0^2)^2] \quad (83)$$

Now we rescale the field, $H \to e^{kr_c\pi}H$ and substitute it in 83 to have an effective action given by

$$S_{vis} \supset \int d^4x\sqrt{-\bar{g}}[\bar{g}^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - e^{-2kr_c\pi}\nu_0^2)^2] \quad (84)$$

Clearly we can see the symmetry breaking scale ν_0 is exponentially suppressed to give an effective scale ν given by

$$\nu = e^{-kr_c\pi}\nu_0 \quad (85)$$

In fact any mass parameter m_0 in the fundamental higher dimensional theory gets exponentially suppressed on visible 3-brane to give rise to a physical mass given by

$$m = e^{-kr_c\pi}m_0 \quad (86)$$

when measured with the metric $\bar{g}_{\mu\nu}$, which is the metric that appears in the effective Einstein action 78.

5.4.1 Weak Scale from Planck Scale

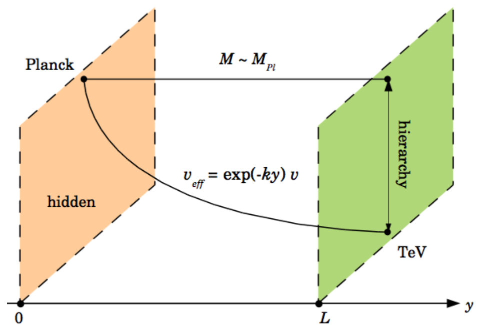

Figure 6: *Generation of Exponential Hierarchy[5]*

If $e^{kr_c\pi}$ is $O(10^{15})$ we could easily generate $\nu \simeq 1TeV$ from fundamental mass parameters not far from the Planck scale, $10^{19}GeV$. Note because of the exponential function we do not require large hierarchies between fundamental parameters $\nu_0, k, M, \mu_c = \frac{1}{r_c}$; in fact $kr_c = \frac{k}{\mu_c} \simeq 10$ will suffice. **Hence we do not have any Disguised Hierarchy**.

6 Radius Stabilation[5, 6]

Untill now the radius of extra dimension has been treated as a parameter which we we set in a fashion which solves the hierarchy problem. However, such a degree of field implies the existence of a massless scalar field,named **Radion** in effective theory corresponding to the fluctuation of radius along extra dimension which in turn would imply the existence of fifth force and violation of Equivalence principle. Therefore, to sustain the Randall-Sundrum model, the radion has to obtain a mass, i.e. it has to be stabilized.The way to do it is to invoke **Goldberger-Wise mechanism**[6].

Note,the Poincare invariance along the 3+1 D hypersurface restricts the metric to be of following form

$$ds^2 = G_{MN}dx^M dx^N = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{87}$$

We introduce a massive scalar field Φ with potential $V(\Phi)$ in the bulk and add $V_1(\Phi)$ and $V_2(\Phi)$ to the branes.The action is given by (Let $y = r_c\phi$)

$$S = \int d^4xdy\sqrt{-G}2M^3R+G^{MN}\frac{1}{2}\partial_M\phi\partial_N\phi-V(\phi)-V_1(\Phi)\delta(y)-V_2(\Phi)\delta(y-L) \tag{88}$$

Varying the action 88 we obtain the equation of motion

$$\frac{1}{\sqrt{-G}}\partial_M\sqrt{-G}G^{MN}\partial_N\Phi = -\partial_\Phi\left(V + V_1\delta(y) + V_2\delta(y-L)\right) \tag{89}$$

Assuming the scalar field only depends on extra dimensional coordinate 44 component will only contribute and we have

$$\Phi'' - 4A'\Phi' = -\partial_\Phi\left(V + V_1\delta(y) + V_2\delta(y-L)\right) \tag{90}$$

On the other hand the 55 component of Einstein field equation i.e $R_{55} - \frac{R}{2}G_{55} = \kappa^2 T_{55}$ yields

$$6(A')^2 = \kappa^2\left(\frac{1}{2}(\Phi')^2 - V(\Phi)\right) \tag{91}$$

which can be reacst in the following form:

$$V(\Phi) = \frac{1}{2}(\Phi')^2 - \frac{6}{\kappa^2}(A')^2 \tag{92}$$

Lets assume the following form of the potential:

$$V(\Phi) = \frac{1}{8}(\partial_\Phi W(\Phi))^2 - \frac{\kappa^2}{6}W^2 \tag{93}$$

Comparing 93 with 92 we obtain following set of equations

$$\Phi' = \frac{1}{2}\partial_\Phi W(\Phi) \tag{94}$$

$$A' = \frac{\kappa^2}{6}W(\Phi) \tag{95}$$

We want our potential to contain a mass term and a constant cosmological constant term. So we take the following ansatz:

$$W(\Phi) = \frac{6k}{\kappa^2} - u\Phi^2 \tag{96}$$

Integrating the first equation of 94 we obtain

$$\Phi = \Phi_P e^{-uy} \tag{97}$$

On TeV brane we have $\Phi_T = \Phi_P e^{-uL}$. We can invert this to obtain

$$L = \frac{ln(\frac{\Phi_P}{\Phi_T})}{u} \tag{98}$$

Thus the value of the radius is determined by equation of motion of scalar field having mass u.

7 Conclusion

The consequences of presence of extra dimensions are far reaching. Although the original idea of Kaluza and Klein to use the extra dimension to unify gravity and electromagnetism turned out to be wrong, the fascination for extra dimension remains alive and later expolited in ADD model and RS model to find a solution to Hierarchy problem.In fact the RS model has profound phenemenological implication i.e exponential generation of hierarchy.The masses in 4 dimensional effective theory depends on the background metric in such a fashion we do not need to invoke large compactification volume (what we need to do in ADD) to achieve the desired hierarchy.Unlike ADD there is no disguised hierarchy between fundamental parameters and the number of extra dimension needed is one in RS as opposed to $n \geq 2$ in ADD. Last but not the least, the solution via RS model needs experimental verication. Hence had it been correct, LHC should come up with rich evidences.

8 Acknowledgement

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