

Simple Harmonic Motion: Geometrical Solutions of Equations of Motions

Sridip Pal*, Soubhik Kumar[†]

Department of Physical Sciences, IISER-Kolkata

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*sridippaliiser@gmail.com

[†]09soubhik@gmail.com

Abstract

Conventionally while we talk about geometry associated with a simple harmonic oscillator(hereafter denoted by,“SHO”) we draw a circle with a radius equal to the amplitude of SHO and imagine a particle moving along the perimeter with a frequency same as that of SHO and then the x coordinate of the position of the particle gives the value of the displacement of the SHO. Here we discuss another kind of diagram (polar plot) depicting SHO and try to get some idea about boundary and initial value problem.¹

1 Introduction

SHO's are ubiquitous in physics. Any potential admitting stable equilibrium can be expanded as a Taylor Series around the point of equilibrium and can be shown to have a Simple Harmonic Oscillatory behaviour for small oscillation around that point. On the other hand, sometimes geometry helps us visualize certain things compared to the algebra involved in the problem. The equation of motion governing the motion of SHO is:

$$\frac{d^2r}{dt^2} = -\omega^2 r \quad (1)$$

where ω is the natural frequency of the oscillator, t is time of observation and r is displacement from origin i.e point of stable equilibrium. The solution to this equation is,

$$r = a \cos(\omega t - \phi) \quad (2)$$

where a and ϕ are respectively the amplitude and the initial phase, determined from given initial or boundary conditions.

First, justification is given why the diagram in Figure 1, what we will be dealing with, depicts SHO. Then, using this diagram, we solve a boundary value problem in which position of the SHO is specified to be x_1, x_2 at time instances respectively t_1, t_2 , and we are required to find the amplitude and initial phase of the oscillator. We also do the same for an initial value problem, where at one instant of time, position and velocity of the oscillator is given.

Of course the boundary value problem can be solved algebraically in a straight forward manner. The geometry supplies us with a new way of obtaining the familiar results.

2 The Diagram

In Figure 1, two circles, of same radius and touching each other externally, are drawn. Their radius and orientations are to be specified later.

We claim that the following Figure 1 represents motion of a SHO in the sense that if a point is moving along this curve then its distance from the origin vary with time in the same way as the distance(not, displacement) of a SHO from the origin. We also identify the polar angle the point makes with X-axis to be ωt

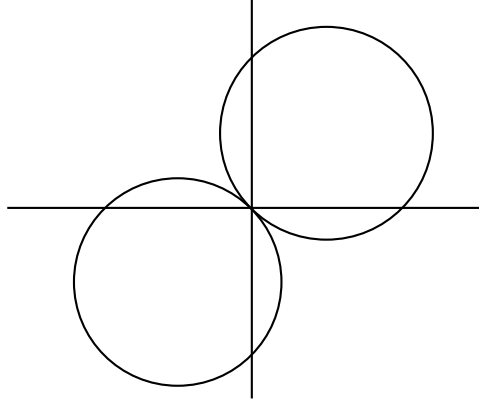


Figure 1: The Solution Circles

2.1 Proof of the Claim

Note, to prove the claim we only need one circle.[see Figure 2] Let O' be the centre of the circle, with OD being the diameter whose length is a . Draw a chord OA . Let, OA be r . Suppose OA makes an angle θ with the diameter. So, $OA = a \cos(\theta)$ since, $\angle OAD$ is $\pi/2$. Since, $r = a \cos \theta$, so (1) is satisfied. And thus our claim is proved.

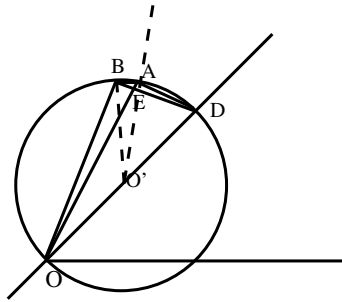


Figure 2: Geometric Solution to Equation of Motion

3 Boundary Value Problem

Now we come to the boundary value problem. Suppose, we are given positions of the oscillators at two time instants, $x(t_1) = x_1$ and $x(t_2) = x_2$ i.e. in a diagram like Figure 3, two lines from the origin OA and OB having lengths $|x_1|$ and $|x_2|$ respectively, are given(Figure 3). So as per previous convention, $\angle AOB = \omega(t_2 - t_1)$. Now,

1. If $\angle AOB < \pi$ (in anticlockwise sense), then draw the circumcircle of $\triangle AOB$.
2. If $\angle AOB > \pi$ (in anticlockwise sense), then reflect OB through the origin and define the reflected line as OB' , now we have made sure that $\angle AOB < \pi$, in fact $\angle AOB = \omega(t_2 - t_1) - \pi$, now we can draw the circumcircle of $\triangle AOB'$.¹

3.1 Finding the Amplitude and Initial Phase

Reflect this circle about origin to get a pair of externally touching circles and note OD' and OB' are the reflections of OD and OB respectively. Let $\angle AOB = \alpha$. We note that the diameter, OD of the circle is amplitude; and the angle it makes with the X axis is minus of the initial phase ϕ . First

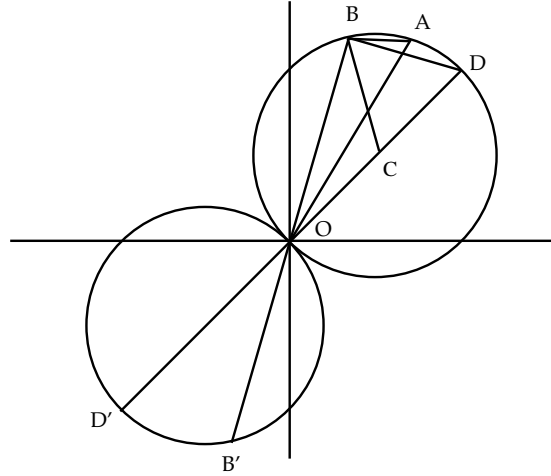


Figure 3: Boundary Value Problem

assume both of x_1 and x_2 are positive and then $\alpha = \omega(t_2 - t_1)$. Later we relax this restriction. Join points to get AB , BD . Since diameter is the biggest of all the chords, so the diameter OD is the amplitude of oscillation. We note that,

$$\angle BDO = \angle BAO \quad (3)$$

¹The case $\angle AOB = \pi$ is discussed later.

since the angles are circumferential and generated by the same chord OB . Applying sin rule[2] on $\triangle AOB$, [Figure 3]

$$\frac{AB}{\sin \alpha} = \frac{OB}{\sin \angle BAO}. \quad (4)$$

Combining we get,

$$\frac{AB}{\sin \alpha} = \frac{OB}{\sin \angle BDO}. \quad (5)$$

As OD is diameter so, $\angle OBD = \pi/2$, and sin rule[2] on $\triangle ODB$ implies:

$$\frac{OB}{\sin \angle BDO} = OD \quad (6)$$

$$\Rightarrow \frac{AB}{\sin \alpha} = OD \quad (7)$$

where the last equation follows using (5). From the triangle $\triangle AOB$ using cosine rule[2] we get:

$$AB^2 = x_1^2 + x_2^2 - 2x_1x_2 \cos \alpha \quad (8)$$

From there we come up with:

$$OD^2 = \frac{x_1^2 + x_2^2 - 2x_1x_2 \cos \alpha}{\sin^2 \alpha} \quad (9)$$

OD gives the amplitude of the SHO. Now, if x_1 and x_2 have opposite signs then $\omega(t_2 - t_1) > \pi$ and a reflection of OB is required, and then $\alpha = \omega(t_2 - t_1) - \pi$; so altogether the form of (8), (9) is retained if we redefine $\alpha = \omega(t_2 - t_1) - \pi$. And lastly if both of x_1 and x_2 are negative the form is same as (8), (9) without any sort of redefinition of α .

So in general, amplitude a is,

$$a = \sqrt{\frac{x_1^2 + x_2^2 - 2x_1x_2 \cos(\omega(t_2 - t_1))}{\sin^2(\omega(t_2 - t_1))}} \quad (10)$$

To know ϕ we are required to measure the angle, diameter makes with the X axis. From the figure it is clear:

$$\phi = \omega t_1 - \arccos \frac{x_1}{a} \quad (11)$$

So we have the complete solution with amplitude and initial phase.

3.2 Uniqueness of Solution

The uniqueness of the solution lies in the fact whether you can draw those circles uniquely or not, given the boundary conditions.

3.2.1 Case:1

$$\omega(t_2 - t_1) = 2n\pi \quad (12)$$

It means, we are specifying the position after integral time period. So for $x_1 \neq x_2$ we do not have any solution at all. If they are equal, it means OA and OB are identical, and the problem can not admit a unique solution as we require at least three non collinear points to construct a circle uniquely. With 2 points (including origin) we can have infinitely many circles each corresponding to a solution satisfying given boundary condition.

3.2.2 Case:2

$$\omega(t_2 - t_1) = (2n + 1)\pi \quad (13)$$

If the boundary condition does not satisfy $x_1 = -x_2$, the system does not admit any solution. Else OA and OB are diametrically opposite, and $\angle AOB = \pi$. Here also we can not draw any unique externally touching circle as the three points including origin is collinear. So we can have infinitely many circles each corresponding to a solution satisfying boundary condition.

3.2.3 Case:3

$$x_1 \text{ or } x_2 = 0 \quad (14)$$

In that case we do not have three non collinear points. But we can still come up with a unique solution (satisfying boundary condition) by drawing a circle with diameter of **nonzero** $\frac{x_i}{\cos(\omega t_i - \phi)}$ and then reflecting it about origin.

3.2.4 Case:4

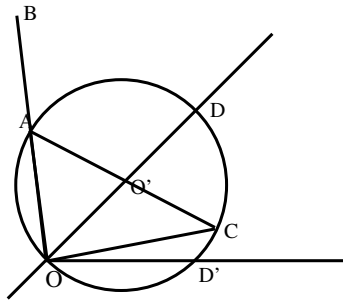
$$x_1 \text{ and } x_2 = 0 \quad (15)$$

We have infinitely many solution provided $\omega(t_2 - t_1) = n\pi$, with one solution corresponding to the situation that the particle does not oscillate at all.

4 Initial Value Problem

First of all, note the solution to SHO (2) implies, $\frac{dr}{d\theta} = -a \sin \theta$ where $\theta = \omega t - \phi$. Geometrically, what it means is the velocity vector can be obtained by rotating the radius vector by $\pi/2$ in counterclockwise sense and multiplying it by ω . In case of initial value problem we are given position and velocity at one instant of time, so if we divide the magnitude of velocity vector by ω we essentially obtain position vector after a quarter time period i.e $\pi/2$,

Let us consider the following Figure 4 with $OA = x(0) = x_1$ and $OB = v(0) = v_1$, where x_1 and v_1 are initial displacement and initial velocity of the SHO. Rotate OB clockwise by an angle of $\frac{\pi}{2}$. Cut OC such that $\omega \times OC = OB$. Now draw the circumcircle of $\triangle AOC$. The diameter of the circle will give amplitude of oscillation while the angle it makes with X axis will give the initial phase ϕ . Since our problem is now reduced to a boundary



value problem as we previously dealt with, only initial value problem fixes α ($\angle AOC$) [Figure 4] equals to $\pi/2$.

$$OD^2 = \frac{x_1^2 + \frac{v_1^2}{\omega^2}}{\sin^2 \frac{\pi}{2}} \quad (16)$$

$$OD^2 = x_1^2 + \frac{v_1^2}{\omega^2} \quad (17)$$

$$r(t) = OD \cos(\omega t - \arccos \frac{x_1}{OD}) \quad (18)$$

4.1 Uniqueness of Solution

$$x(0) \text{ or } v(0) = 0 \quad (19)$$

In the same way as discussed in case:3.2.3 of boundary value (B.V) problem, we draw a unique circle of diameter of $x(0)$ or $\frac{v(0)}{\omega}$ whichever is non-zero.

$$x(0) \text{ and } v(0) = 0 \quad (20)$$

Quite contrary to the case:3.2.4 of the B.V problem here we have only one solution that the particle does not oscillate at all because in whatever way we draw the circle we have to traverse an angle of $n\pi$ [see the Figure 4] to come to 0,starting from 0. But we know,the angle between OA and OC is $\frac{\pi}{2}$.So it can only be possible when the circles degenerate into a point of origin i.e no oscillation.

5 Generalization

Although the above argument was entirely for SHO, but one can generalize it to any symmetric potential where oscillation is taking place about the point on symmetry axis. For those cases, instead of circles in Figure 1, one gets other geometrical shapes. But one fact remains true, that the same two lobes are obtained, and as a whole the figure remains symmetric about $y = x$ line. Now, because of this, if the values after half a time period are specified then, one can reflect the point about the origin, and then the two points coincide. So, we can be sure that no unique solution to the problem is possible, since with just origin and one point one draw any curves. Now, from Newton's law, we know we require just two points to draw a unique curve (except for special initial or boundary values), but at this moment it is not clear how can we argue about this necessity purely from geometry, for circles it was possible since we need just three points in general to draw a curve.

6 Conclusion

It is satisfying that simple geometrical considerations solves the aforementioned problems. One usually uses algebraic methods to tackle these problems, but the uniqueness and existence of solutions is very transparent in this approach. These type of diagrams can give us an intuitive feeling about the system without any direct dependence on algebraic machinery. The whole scenario hinges on the fact (2) gives a diagram like mentioned above when plotted in polar co-ordinates. So any system (like Damped, Driven Harmonic Oscillator) whose solution can be reduced to this form (2) by proper substitution, we can employ this geometric method to yield solution to boundary and initial value problems.

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