IISER KOLKATA WINTER SCHOOL ON NUMBER THEORY : *WARING'S PROBLEM*

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Abstract

In the year 1770 **Edward Waring** conjectured : *Every natural number can be written as sum of 4 squares, as the sum of 9 cubes, as the sum of 19 4th power and so on....* To rephrase for every given integer $k \geq 2, \exists g(k) = g$ such that every natural number can be written as sum g kth power.In the following lecture note we will go through a complete proof of Waring's problem showing the existence of g=g(k) by a method proposed by *Linnik* and further improved by *Hua*.Along with we will be introduced to concept of *Schnirelman Density* and briefly look over the history of the problem.

1 History

In the same year the existence of $g(k)$ is conjectured for every integer $k \geq 2$, *Lagrange* managed to show $g(2) = 4$.Then it took more than 100 years to prove the conjecture made by Waring.The proof came in the year 1909 by *Hilbert*. In that period of time g(3) was proved to be 9 due to work of *Wiefric and Kempner*.Unfortunately,Hilbert's proof was nonconstructive.So people tried to cook up a formula, at least an upper bound for g(k).In the mean time genesis of *circle method* by *Ramanujan* and its application to attack the Waring's problem by *Hardy, Littlewood* added to momentum.In 1946 *Linnik* provided with an elementary approach using idea of *Schinirelman*.Later the method was improved by *Hua, Nesterenko, Rammurty*.In 1940 *Pillai* proved g(6)=73 and in 1964 *Chen* proved g(5)=37 while g(4) was proved to be 19 only in the year 1986 by *Balsubhramanium and Deshouillers*.It is interesting to note that *J.A.Euler* conjectured long back in the year 1772 $g(k) = 2^k + [\frac{3^k}{2^k}] - 2$. Pillai managed to show Euler's conjecture is true provided a certain condition holds,but noboby knows till today whether the condition holds for all positive integers.So there is a big hole and we hope someone with enough brilliance with come up with an excellent trick to bridge the gap.

2 Schnirelman Density

For clarity we announce the sets we will deal with in subsequent sections are subset of nonnegative integers unless specified.

Let $A \subseteq N$..Define $A(n) = |\{a \in A : a \leq n\}|$.

Now *Schnirelman Density* of set A is defined to be $\delta(A) = \inf_{n \geq 1} \frac{A(n)}{n}$. Readers are left to prove that Schnirelman Density of any set is 1 iff the set is identical to the set of natural numbers.

2.1 Theorem 1(Schnirelman,1936)

$\delta(A + B) \geq \delta(A) + \delta(B) - \delta(A)\delta(B)$ where $A + B = \{a + b : a \in A, b \in B\}$ including the empty choice i.e A and B both are subset of $A + B$.

Proof:

Let $A = \{a_1, a_2, ...a_r, a_{r+1}, ...\}$ with $a_{i+1} > a_i$ and $a_r \leq n$, $a_{r+1} > n$.

$\Rightarrow A(n) = r$

Let $B = \{b_1, b_2, ...b_s, b_{s+1}, ...\}$ with $b_{i+1} > b_i$ and $b_s \leq n$, $b_{s+1} > n$.

$\Rightarrow B(n) = s$

Now we will count in a simple manner.We would find out how many numbers of the set $A+B$ there are between consecutive a_i's. At least we will try to get a lower bound.Now the numbers in B which is less than $(a_{i+1}-a_i-1)$ can be added to a_i to get a number which is less than a_{i+1}.So we get a lower bound i.e $B(a_{i+1}-a_i-1)$.We have r elements from set A and from B we make an estimate.(Readers are urged to find out why the exact equality does not hold.Hint:there may be more elements in $A+B$).

$$(A+B)(n) \geq A(n)+B(a_1-1)+\sum_{1\leq i\leq r-1} B(a_{i+1}-a_i-1)+B(n-a_r) \quad (1)$$

Since $A(n) \geq \delta(A)n$

$$\Rightarrow L.H.S \geq A(n)+\delta(B)[(a_1-1)+\sum_{1\leq i\leq r-1}(a_{i+1}-a_i-1)+(n-a_r)] \quad (2)$$
$$= A(n)+\delta(B)(n-r) \quad (3)$$
$$= A(n)+\delta(B)(n-A(n)) \quad (4)$$
$$= A(n)(1-\delta(B))+n\delta(B) \quad (5)$$
$$\geq \delta(A)n(1-\delta(B))+n\delta(B) \quad (6)$$
$$= n[\delta(A)+\delta(B)-\delta(A)\delta(B)] \quad (7)$$
$$\Rightarrow \frac{(A+B)(n)}{n} \geq [\delta(A)+\delta(B)-\delta(A)\delta(B)] \quad (8)$$

So it follows

$$\delta(A+B) \geq [\delta(A)+\delta(B)-\delta(A)\delta(B)] \quad (9)$$

By induction we can easily prove

$$\delta(\sum_{i\leq t} A_i) \geq [1-\prod_{i\leq t}(1-\delta(A_i)] \quad (10)$$

2.2 Theorem 2 on Schnirelman Density

Statement:
If $\delta(A)$ is positive,then there exists a positive integer m such that $\delta(mA)=1$ where mA means A added m times.
Proof:
We will first show there exist m_0 such that $\delta(m_0A) > \frac{1}{2}$
We know

$$\delta(tA) \geq 1-(1-\delta(A))^t \quad (11)$$

If $\delta(A)=1$ we are done proving the theorem.If not then choose t such that $(1-\delta(A))^t$ becomes less than $\frac{1}{2}$ exploiting the fact $\delta(A)$ lies between 0 and 1.Let $t=m_0$ and we are done proving the existence of a set with Schnirelman density greater than $\frac{1}{2}$.
Let $m_0A=B$.We will now show $\delta(2B)=1$
Now,let us introduce 2 sets:

$$|W=\{b : b\in B, b\leq n\}| > \frac{n}{2} \quad (12)$$

$$|W' = \{n - b : b \in B, b \leq n\}| > \frac{n}{2} \tag{13}$$

As the number of elements less than or equal to n cant be greater than n we conclude W and W' cant be disjoint due to PHP.There exists b_0 and b'_0 such that $n = b_0 + b'_0$ proving $\delta(2B) = 1$

3 Attacking Waring's Problem via Schnirelman Density

We will rephrase the Waring's problem in the terms of Schnirelman Density.We introduce a set in following manner:

For every k $A_k = \{x^k : x \in N \cup \{0\}, k \in N\}$

If we can show for some g, $\delta(gA_k) > 0$,we are done asking the theorem 2 by Schnirelman.

Define $r_{g,k}(n) = |R_n = \{(x_1, x_2, x_3, ...x_g) : \sum_{i=1}^{g} x_i^g = n\}|$.

Let $g(k, n) = \sum_{m \leq n} r_{g,k}(m)$.We'll now use a lemma proved by Linnik many years back.

Linnik's Lemma:

For every given k,ther exists g such that $r_{g,k}(n) \leq c(k)n^{\frac{g}{k}-1}$.

Using the lemma we can cook up an upper bound for $g(k, n)$.

$$g(k, n) \leq \sum_{m \leq n} m^{\frac{g}{k}-1} \tag{14}$$

$$\Rightarrow g(k, n) \leq c(k)n^{\frac{g}{k}-1} \sum_{m \leq n} 1 \tag{15}$$

We realise the summands of $g(k, n)$ contribrute iff m can be written as a sum of g kth powers. So it follows

$$g(k, n) \leq c(k)n^{\frac{g}{k}-1}(gA)(n) \tag{16}$$

Well let us find a lower bound by building up the set $\cup_{m \leq n} R_m$ as the cardinality of this set equals $g(k, n)$,R_m's being disjoint.We can vary x_i's from 0 to $(\frac{n}{g})^{\frac{1}{k}}$ to get a sum less than or equal to n.So we have atleast $(\frac{n}{g})^{\frac{g}{k}}$ choices. $\Rightarrow g(k, n) \geq (\frac{n}{g})^{\frac{g}{k}}$

Combining 2 inequality we get

$$(gA)(n) \geq c'(k)n > 0 \tag{17}$$

Bingo!!!!!We are done using the theorem 2 by Schnirelman. We managed to show the existence of g=g(k) for every k i.e every natural number can be written as a sum of g kth powers provided we prove Linnik's lemma.

4 From Hua to Linnik

To prove Linnik's lemma we will use another beautiful result due to *Hua*.

Statement:

For every k there exists $g \in N$ such that

$$\int_0^1 |\sum_{0\le x\le P} \exp(2\pi\imath x^k\alpha)|^g d\alpha \le c(k)P^{g-k} \tag{18}$$

Proof:

$$(\sum_{0\le x\le P} \exp(2\pi\imath x^k\alpha))^g = \sum_{x_i=0}^{P} \exp(2\pi\imath(\sum_{i=0}^{g} x_i^k)\alpha) \tag{19}$$

$$\Rightarrow (\sum_{0\le x\le P} \exp(2\pi\imath x^k\alpha))^g = \sum_{m\ge 0} c_m \exp(2\pi\imath m\alpha) \tag{20}$$

By fourier trick we can easily find out c_m's. [*If readers are unfamiliar with fourier trick,then they are urged to prove the following problem:Let* $f(\alpha) = \sum_{m\ge 0} c_m \exp(2\pi\imath m\alpha)$. *Then* $c_m = \int_0^1 f(\alpha)\exp(-2\pi\imath m\alpha)d\alpha$]

$$c_n = \int_0^1 \sum_{0\le x\le P} \exp(2\pi\imath x^k\alpha)^g \exp(-2\pi\imath n\alpha)d\alpha \tag{21}$$

Now observe c_n exactly counts the no of ways n can be written as a sum of g kth power i.e $r_{g,k}(n) = c_n$ if we let $P = n^{\frac{1}{k}}$.(Readers are left to find out what would happen if we let $P < n^{\frac{1}{k}}$).

$$r_{g,k}(n) = |r_{g,k}(n)| = |\int_0^1 \sum_{0\le x\le P} \exp(2\pi\imath x^k\alpha)^g \exp(-2\pi\imath n\alpha)d\alpha| \tag{22}$$

$$\Rightarrow r_{g,k}(n) \le \int_0^1 |\sum_{0\le x\le P} \exp(2\pi\imath x^k\alpha)|^g |\exp(-2\pi\imath n\alpha)|d\alpha \tag{23}$$

But $|\exp(\imath\theta)| = 1$ for $\theta \in \Re$. Now using Hua's Lemma we can conclude

$$r_{g,k}(n) \le c(k)n^{\frac{g}{k}-k} \tag{24}$$

Hence Linnik's Lemma is proved provided we have a proof of Hua's Lemma. Notice how we are approaching to solve main problem.We are cooking up proofs one after one,using lemmas to reduce the problem to a one which can be solved in an elementary method.It should be emphasised that elementary method may not be easy rather tricky but all it uses are very elementary concepts.So our journey from Linnik to Hua is a step forward for something elementary.

5 Hua's Lemma and its Proof

Now we will state the generalised version of Hua's lemma:

Let

$$f(x) = \sum_{n=0}^{k} a_n x^n \in Z[x] \tag{25}$$

with $a_{k-m} = O(P^m)$ for all m where O is definded in following way: $a = O(k) \iff |a| \leq ck$ for some constant c.It is a neater way to keep track of the order of the numbers without worrying much about constant factors.Readers can easily check out the following properties of O which will be in use later.

$$a = O(0) \Rightarrow a = 0 \tag{26}$$

$$a = O(x)O(y) \Rightarrow a = O(xy) \tag{27}$$

$$a = min\{O(x), O(y)\} \Rightarrow a = O(min\{x, y\}) \tag{28}$$

Hua's lemma says:

$$\int_0^1 |\sum_{0\leq x\leq P} \exp{(2\pi\imath f(x)\alpha)}|^{8^{k-1}} d\alpha = O(P^{8^{k-1}}) \tag{29}$$

Proof:

To prove the Hua's lemma we will make use of following lemma which is quite interesting in its own right. Let $q(n)=$ Number of integer solution of $x_1y_1 + x_2y_2 = n$ with $|x_i| \leq X$ and $|y_i| \leq Y$.

Our claim is:

$$q(0) = O((XY)^{\frac{3}{2}}) \tag{30}$$

$$q(n \neq 0) = O(XY \sum_{ds=n} \frac{1}{d}) \tag{31}$$

Proof of the claim:

Case 1.$n = 0$

We will count in two ways to estimate the numbe of solutions.First let us choose x_1 in $O(X)$ ways and so the x_2. Now choose y_1 in $O(Y)$ ways.y_2 gets fixed automatically by the equation.In this way of counting we will certainly overshoot $q(0)$ because y_2 has to be an integer.So the number of choices:

$O(X)O(X)O(Y) = O(X^2Y)$

Now by same counting trick choosing y_i's first we get the number of choices: $O(XY^2)$

Combining two bounds we conclude

$$q(0) = min\{O(X^2Y), O(XY^2)\} \tag{32}$$

$$\Rightarrow q(0) = O(min\{X^2Y, XY^2\}) \tag{33}$$

$$\Rightarrow q(0) = O((XY)^{\frac{3}{2}}) \tag{34}$$

The last line follows from following observation $min(|A|, |B|) = \sqrt{|A||B|}$.

Case 2:$n \neq 0$

Choose x_1, x_2 such that $\gcd{(x_1, x_2)} = 1$. Let y_{10} and y_{20} are solution.Then all other solution are given by

$y_1 = y_{10} + tx_2$

$y_2 = y_{20} - tx_1$

Now x_1 and x_2 both cant be 0 as $n \neq 0$.Without loss of genarality assume $x_1 \neq 0$. As $|y_i| \leq Y$,number of choices of t equals $O(\frac{Y}{|x_1|})$.

Therefore the number of solution

$$= O(\sum_{|x_1|\leq X}^{x_1\neq 0} \sum_{|x_2|\leq|x_1|} \frac{ky}{|x_1|}) \tag{35}$$

$$\Rightarrow q(n \neq 0) = O(XY) \tag{36}$$

Now let $(x_1, x_2) = d$ and $\frac{x_i}{d} = r_i$.

$$r_1y_1 + r_2y_2 = \frac{n}{d} \ with \ (r_1, r_2) = 1 \ and \ r_i = O(\frac{X}{d}) \tag{37}$$

$$\sharp \ solution = O(\frac{XY}{d}) \tag{38}$$

Now the solution of the equation exists iff $ds = n$.(Readers are left to prove this elementary theorem).So

$$q(n \neq 0) = O(XY \sum_{ds=n} \frac{1}{d}) \tag{39}$$

5.1 Hua's Lemma for k=2

We need to show

$$\int_0^1 |\sum_{x=0}^{P} \exp{(2\pi\imath f(x)\alpha)}|^8 d\alpha \leq P^6 \tag{40}$$

where $f(x) = a_2x^2 + a_1x + a_0$ with $a_i = O(P^{2-i})$

$$L.H.S = \int_0^1 (\sum_{x=0}^{P} \exp{(2\pi\imath f(x)\alpha)})^4 (\sum_{x=0}^{P} \exp{(-2\pi\imath f(x)\alpha)})^4 d\alpha \tag{41}$$

$$= \int_0^1 \sum_{x_i=0, 1\leq i\leq 4}^{P} \exp{(2\pi\imath \sum_{i=0}^{4} (f(x_i) - f(y_i))\alpha)} d\alpha \tag{42}$$

$$= \ Number \ of \ solution \ of \ following \ \sum_{i=1}^{4} f(x_i) = \sum_{i=0}^{4} f(y_i) \tag{43}$$

Since for if the sums are not equal then the integral contribrutes nothing upon integrating in the desired interval.When they are equal they contribute exactly 1.So the integral is number of solution of following equation:

$$f(x_1) - f(y_1) + f(x_2) - f(y_2) = f(y_3) - f(x_3) + f(y_4) - f(x_4) \tag{44}$$

$$Now \ f(x_i) - f(y_i) = (x_i - y_i)[a_2(x_i + y_i) + a_1] \tag{45}$$

$$Let \ us \ introduce \ z_i = x_i - y_i, w_i = [a_2(x_i + y_i) + a_1] \tag{46}$$

$$Hence \ L.H.S = \sharp \ solution \ of \ z_1w_1 + z_2w_2 = z_3w_3 + z_4w_4 \tag{47}$$

Where $z_i, w_i = O(P)$

Hence $z_1w_1 + z_2w_2 = O(P^2)$.So the integral i.e number of solution is(for $n = z_1w_1 + z_2w_2 = z_3w_3 + z_4w_4$ we have $q(n)^2$ solutions)

$$\sum_{|n|\leq cP^2} q(n)^2 \tag{48}$$

$$= q(0)^2 + \sum_{|n|\leq cP^2}^{n\neq 0} q(n)^2 \tag{49}$$

$$= O((P^2)^{\frac{3}{2}})^2 + \sum_{|n|\leq cP^2}^{n\neq 0} O(P^2 \sum_{ds=n} \frac{1}{d})^2 \tag{50}$$

$$= O(P^6) + \sum_{|n|\le cP^2}^{n\neq 0} O(P^4 \sum_{d_1p=n,d_2q=n} \frac{1}{d_1d_2}) \tag{51}$$

Here we have used the previously proved lemma in 4th line concerning number of solutions. Now to estimate the 2nd term we will change order of summation.By careful observation we can write:

$$\sum_{|n|\le cP^2}^{n\neq 0} P^4 \sum_{d_1p=n,d_2q=n} \frac{1}{d_1d_2} \le P^4 \sum_{d_i\le cP^2} \sum_{|n|\le cP^2}^{[d_1,d_2]s=n} \frac{1}{d_1d_2} \tag{52}$$

where$[d_1, d_2]$ implies lcm of them.

In L.H.S a particular summand occurs exactly the times d_1, d_2 occurs as a divisor of n with $|n| \le cP^2$ which is clearly $[\frac{P^2}{[d_1,d_2]}]$ which is less than or equal to $\frac{P^2}{[d_1,d_2]}$.Knowing $[d_1, d_2](d_1, d_2) = d_1d_2$, we can write-

$$\sum_{d_i\le P^2} \frac{1}{[d_1, d_2]d_1d_2} = \sum_{d_i\le P^2} \frac{\gcd(d_1, d_2)}{(d_1d_2)^2} \tag{53}$$

$$n = \sum_{ds=n} \phi(d) \tag{54}$$

where $\phi(n) = |\{a \le n : \gcd(a, n) = 1\}|$

(Readers are left to prove equation(51).A sketchy outline of the proof is given:

1. Let $S_d = \{1 \le a \le n : (a, n) = d\}$.Prove $\cup_{ds=n} S_d = \{1, 2, 3, ...n\}$ and the union is disjoint.
2. Construct a bijection between S_d and $\{1 \le b \le \frac{n}{d} : (b, \frac{n}{d}) = 1\}$ to deduce $|S_d| = \phi(\frac{n}{d})$
3. Deduce $\sum_{ds=n} \phi(\frac{n}{d}) = \sum_{ds=n} \phi(d) = n$

where $(a, n) = \gcd(a, n)$. This proves the result.)

$$\sum_{d_i\le P^2} \frac{\gcd(d_1, d_2)}{(d_1d_2)^2} = \sum_{d_i\le P^2} \frac{1}{(d_1d_2)^2} \sum_{\delta k=(d_1,d_2)} \phi(\delta) \tag{55}$$

$$\Rightarrow L.H.S \le \sum_{\delta\le P^2} \phi(\delta) \sum_{\delta k=(d_1,d_2)} \frac{1}{(d_1d_2)^2} \tag{56}$$

$$\le \sum_{0}^{\infty} \frac{\phi(\delta)}{\delta^4} \sum_{t_1,t_2=0}^{\infty} \frac{1}{(t_1t_2)^2} \tag{57}$$

$$\le k \tag{58}$$

where k is a constant independent of P.Here we have changed the order of summation and extend the sum to infinity to get an estimate of upper bound.Hence what we get doing all this stuff is as follows

$$\sum_{1\le|n|\le cP^2} P^4 (\sum_{ds=n} \frac{1}{d})^2 = O(P^6) \tag{59}$$

$$\int_0^1 |\sum_{x=0}^{P} \exp(2\pi\imath f(x)\alpha|)|^8 d\alpha = O(P^6) \tag{60}$$

5.2 Hua's Lemma for k >2

Now we will apply induction method to prove the 2nd version of Hua's lemma in general.

Consider $f(x)$ *being a polynomial of degree* k satisfying Hua's lemma's requirement:

$$|\sum_{x=0}^{P} \exp\left(2\pi\imath f(x)\alpha\right)|^2 = \sum_{x_1,x_2=0}^{P} \exp\left[2\pi\imath(f(x_1) - f(x_2))\right]\alpha \tag{61}$$

$$= \sum_{x=0}^{P} \exp\left(-2\pi\imath f(x)\alpha\right) \sum_{h}^{\prime} \exp\left(2\pi\imath f(x+h)\alpha\right) \tag{62}$$

$$= (1+P) + \sum_{h\neq 0}^{\prime} \sum^{\prime} \exp\left(2\pi\imath[f(x+h) - f(x)]\alpha\right) \tag{63}$$

$$= (1+P) + \sum_{h\neq 0}^{\prime} a_h \tag{64}$$

Raising both sides of last equation to the power 8^{k-2} we get

$$|\sum_{x=0}^{P} \exp\left(2\pi\imath f(x)\alpha\right)|^{2.8^{k-2}} = |(1+P) + \sum_{h\neq 0}^{\prime} a_h|^{8^{k-2}} \tag{65}$$

$$\Rightarrow L.H.S \ll (P^{8^{k-2}} + |\sum_{h\neq 0} a_h|^{8^{k-2}}) \tag{66}$$

where $\ll$ means the same as O.

(Readers are left to prove the following result what we have used to get (63) from (62): $|A+B|^{2^k} \leq c_k(|A|^{2^k} + |B|^{2^k})$ where $c_k = 2^{2^k-1}$.Hint:Use induction along with Cauchy-Schwarz inequality.)

Suppose $|\sum_{h\neq 0} a_h| \ll P$.

Raising (63) to the power 4,we get

$$|\sum_{x=0}^{P} \exp(2\pi\imath f(x)\alpha)| \ll P^{8^{k-1}.4} \tag{67}$$

$$\Rightarrow \int_0^1 |\sum_{x=0}^{P} \exp(2\pi\imath f(x)\alpha)| d\alpha \ll P^{8^{k-1}.4} \tag{68}$$

$$\Rightarrow \int_0^1 |\sum_{x=0}^{P} \exp(2\pi\imath f(x)\alpha)| d\alpha \ll P^{8^{k-1}-k} \tag{69}$$

as $8^{k-2}.4 \leq 8^{k-1} - k$.So we are done.But where have we used the induction hypothesis.Wait,we have to use it in case of $|\sum_{h\neq 0} a_h| \geq P$.

Applying Triangle inequality,Cauchy-Schwarz inequality and recaling the fact h varies in an interval of length P we get,

$$|\sum_{h\neq 0} a_h|^{8^{k-2}} = [|\sum_{h\neq 0} a_h.1|^2]^{2^{3(k-2)-1}} \tag{70}$$

$$\leq [\{\sum_{h\neq 0}|a_h|.|1|\}^2]^{2^{3(k-2)-1}} \quad [Triangle\ Inequality] \tag{71}$$

$$\leq [\sum_{h\neq 0} 1^2 \sum_{h\neq 0}|a_h|^2]^{2^{3(k-2)-1}} \quad [Cauchy-Schwarz\ inequality] \tag{72}$$

$$= P^{2^{3(k-2)-1}}[\{\sum_{h\neq 0}|a_h|^2\}^2]^{2^{3(k-2)-2}} \tag{73}$$

Apply again Cauchy-Schwarz Inequality to term containing a_h and go on doing this for $3(k-2)$ times.This yields

$$|\sum_{h\neq 0} a_h|^{8^{k-2}} \leq P^{\sum_{i=0}^{3(k-2)} 2^{3(k-2)-i}} \sum_{h\neq 0}|a_h|^{8^{k-2}} \tag{74}$$

$$= P^{2^{3(k-2)}-1}\sum_{h\neq 0}|a_h|^{8^{k-2}} \tag{75}$$

Now raising to the power 4 yields

$$|\sum_{x=0}^{P}\exp(2\pi\imath f(x)\alpha|^{8k-1} \ll P^{4(2^{3(k-2)}-1)}(\sum_{h\neq 0}|a_h|^{8^{k-2}})^4 \tag{76}$$

$$\int_0^1 |\sum_{x=0}^{P}\exp(2\pi\imath f(x)\alpha|^{8k-1}d\alpha \ll P^{4(2^{3(k-2)}-1)}\int_0^1(\sum_{h\neq 0}|a_h|^{8^{k-2}})^4 d\alpha \tag{77}$$

Let us now estimate the $f(x+h)-f(x)$

$$f(x+h)-f(x) = \sum_{j=0}^{k-1} b_j[(x+h)^j - x^j] \tag{78}$$

$$= \sum_{j=0}^{k} b_j \sum_{i=0}^{j-1} x^i h^{j-i}(j\ Choose\ i) \tag{79}$$

$$= h\phi(x,h) \tag{80}$$

with $\phi(x,h)$ satisfies Hua's lemma's requirement and $h = O(P)$.Readers are urged to check this is indeed the case as a little bit of brainteasing element.Hint: $\phi(x,h)$ is a polynomial of degree $(k-1)$ and recall $b_k h^{k-i-1} = O(P^{(k-1)-i})$.

So what we get at last is

$$P^{4(2^{3(k-2)}-1)}\int_0^1(\sum_{h\neq 0}|a_h|^{8^{k-2}})^4 d\alpha = P^{4(2^{3(k-2)}-1)}A(n_1)A(n_2)\overline{A(n_3)A(n_4)}t \tag{81}$$

$$\Rightarrow \int_0^1 |\sum_{x=0}^{P}\exp(2\pi\imath f(x)\alpha)|^{8k-1}d\alpha \ll P^{4(2^{3(k-2)}-1)}A(n_1)A(n_2)\overline{A(n_3)A(n_4)}t \tag{82}$$

where

$$|a_h|^{8^{k-2}} = \sum_n A(n)\exp(2\pi\imath hn\alpha) \tag{83}$$

$$n_i = \phi(x,h_i) \tag{84}$$

$$t = \int_0^1 \sum_{h_i,n_i}\exp[2\pi\imath(h_1n_1+h_2n_2-h_3n_3-h_4n_4)\alpha]d\alpha \tag{85}$$

Now t=number of solution of $h_1n_1+h_2n_2 = h_3n_3+h_4n_4$ *with* $h_i = O(P),\ n_i = O(P^{k-1})$ because the summands contribute iff the solution exists for the aforesaid equation and the contribution is exactly 1.

Applying fourier trick,we get

$$A(n) = \int_0^1 |a_h|^{8^{k-2}} \exp(-2\pi\imath hn\alpha)d\alpha \tag{86}$$

$$\leq \int_0^1 |a_h|^{8^{k-2}} d\alpha \tag{87}$$

$$\leq P^{8^{k-2}-(k-1)}\ [Induction\ Hypothesis\ applied\ on\ \phi(x,h)] \tag{88}$$

As $h_i = O(P)$ *and* $n_i = O(P^{k-1})$ By same counting algorithm we come up with the fact $t = O((P^{1+(k-1)})^3) = O(P^{3k})$

Summing up all the results obtained we get,

$$\int_0^1 |\sum_{x=0}^{P} \exp(2\pi\imath f(x)\alpha)|^{8^{k-1}} d\alpha \ll P^{4(8^{k-2}-1)} P^{4(8^{k-2}-(k-1))} P^{3k} \tag{89}$$

$$\Rightarrow \int_0^1 |\sum_{x=0}^{P} \exp(2\pi\imath f(x)\alpha)|^{8^{k-1}} d\alpha = O(P^{8^{k-1}-k}) \tag{90}$$

By mathematical induction the result holds for all positive integer k>2.We have atlast reached our goal.We have proved Waring's conjecture. BINGO!!

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