# World Sheet Current and Conservation

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- The term "Conserved Current" is slight misleading, because actually charge is what gets conserved where charge is defined as $\int j^0 dx^D$.

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- Here the conserved charges are $Q_i = \int dx^D j_i^0$.

# Euler Lagrange Equation for Field

$$\frac{\partial \mathcal{L}}{\partial \phi^a} = \partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial_\alpha \phi^a} \right) \tag{2}$$

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Starting from these 2 simple equations we can prove our previous claim of conservation law i.e $\partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \delta \phi^a - \epsilon^i \Lambda_i^\alpha \right) = 0$

# Generalisation

- In the same way as outlined in previous slide, one can show if under any transformations $\delta(\mathcal{L}) = \partial_\alpha(\epsilon^{i_1,i_2,\ldots,i_n}\Lambda^\alpha_{i_1,i_2,\ldots i_n})$ then $\partial_\alpha j^\alpha_{\sigma(i_k)} = 0$ where $\epsilon^{i_1,i_2,\ldots i_n} j^\alpha_{i_1,i_2,\ldots i_n} = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a - \epsilon^{i_1,i_2,\ldots i_n}\Lambda^\alpha_{i_1,i_2,\ldots i_n}$

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- The Energy and Momentum is conserved for free particle.

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- Lorentz invariance demands antisymmetric nature of $\epsilon^{\mu\nu}$

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- In 3+1 Dimension we just assign $L_k = j_{\mu\nu}$.

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- The result is $E = mc^2$.The one of the most powerful equation in physics is really a consequence of LORENTZ SYMMETRY!!

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- We take the previous 2 transformations and look for conserved charges associated with String Motion.!!

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- So these current lives on the world sheet only,not on the whole space-time.
- The index $\mu$, $\nu$ vary over the space time index.So for each such index or index pair we have a conservation law.

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- In case of open but not free string, $p_\mu$ fails to conserve if we consider string only, yet the total momentum of D-brane along with string is conserved.
- Reparameterisation invariance allows us to let $\tau$ be $t$ of any lorentz frame and let that observer to confirm the conservation of momentum carried by the string.

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- $M_{\mu\nu}(\tau)$ can also be shown to be parameterisation invariant.

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