Superconductivity:Macroscopic Quantum Mechanics

Sridip Pal (09MS002)
DPS
IISER Kolkata
QM-3 Term Paper

November 28, 2012

Abstract

The pivotal point of the term paper is to introduce Superconductivity as a **macroscopic manifestation** of Quantum Mechanics.It depicts superconductivity as a **collective** phenomenon. The authors discusses some landmarking events concerning superconductivity viz. Meissner effect in simply and non simply connected topology, the aharanov-Bohm effect, flux quantisation, Josephson junction and its application in Squid. The author also touches the basics of BCS theory to give a microscopic understanding of Superconductivity.

1 Introduction

Superconductivity is one of the major breakthrough in the history of Physics.In the year 1911 just after the refrigeration technique via liquid helium has emerged, H.K.Onnes discovered something capable of carrying current without any resistance when it is cooled down below a certain critical temperature of the order of few Kelvin.Initially it was done with Mercury. Later people started to find metals beyond certain low temperature are capable of being **superconductor**. The year 1933 came with another surprise; Meissner and Oschenfeld observed superconductors are capable of expelling magnetic fields.Within 1950's and 1960's a complete and satisfactory theory of classical superconductor had been revolutionised with the emergence Landau-Gingburg effective theory and most celebrated microscopic theory of superconductor given by Bardeen, Cooper, Schreifer (BCS) in 1957.

In this report we will give a brief overview and basics of BCS theory and cooper pairs in first few sections and try to justify the existence of supercurrent.In section 2, we will briefly say in a simple minded fashion how the cooper pairs are formed and say about instability of Fermi sea against formation of cooper pair when we have attractive interaction potential. Section 3 contains mathematical formulation of BCS ground state and derivation of conjugacy relationship between Number of pairs and phase.In section 4 we depict superconductivity as a macroscopic manifestation of QM with a reinterpretation probability amplitude.Meissner effect in a lump superconductor is detailed in section 5 along with derivation of London and London Equation.In section we explain the famous Aharanov-Bohm effect and flux quantisation in context of Meissner effect in a nonsimply connected topological superconductor.Section 6 illustrates Josephson junction,another remarkable phenomenon concerning superconductor; its DC and AC counterpart. In section 7 we introduce magnetic field in the picture and study Josephson junction coupled with Aharanov-Bohm effect resulting to interference between currents and emergence of SQUID-Superconducting Quantum Interference Device. Section 8 elaborates the DC and RF SQUID with its brief application.Section 9 has been put up for sake of completeness where we discuss the dynamics of superconductivity and compare it to the familiar equation of hydrodynamics and lorentz force.The report concludes with some remarks on what lie ahead.

2 Cooper Pairs and Cooper Instability-a microscopic viewpoint

As promised in the introduction this section will introduce **Cooper Instability** in a simple minded fashion which was first put forward by Cooper in one of his seminal paper and later given a strong platform by Bardeen, Cooper and Schreifer in the name of BCS theory.

The crux of the whole story behind BCS is that electrons are allowed to talk to each other i.e electrons of opposite spin and momenta can essentially pair up (**famous as Cooper pair**) with the help of mediating phonons to form a Bosonic ground state. As we exchange 2 pairs, we are in fact exchanging two electrons, hence the -1 eigenvalue of exchange operator due to Fermi statistics gets multiplied twice to yield 1 eigenvalue which is indeed Bosonic feature.This novel feature makes superconductivity to emerge as a macroscopic realisation of Quantum Mechanics. We can put a macroscopic number of pairs in a single superconducting state due to this bosonic character and it allows to reinterpret the probability amplitude ψ, a point which we will come later in section 4. Now comes the question why such pairing would at all happen. The answer lies in the fact that pairing can lower the energy of the system as shown in BCS.

Before delving into how energy gets lowered lets think of a simple picture of one electron passing through a lattice. As it passes through, it brings about accumulation of positive charge of nuclei which in turn attracts another electron. The subtlety is electronic time scale is larger than the nuclear time scale as nuclei is thousand times heavier than electron. Hence, even after the electron has moved past, the charge density stays and it attracts another electron. Therefore we have the first electron effectively interacting with second one via attractive potential and forming a pair. The second electron is kind of following the first guy. Note the electrons in a pair can really be far apart, spread over a considerable distance, in fact the mean distance between pairs is relatively smaller than the size of a single pair.The mediation of phonon is experimentally confirmed by isotope effect where critical field and temperature varies with isotope i.e nuclear charge.

Lets be bit formal now and we will see how really an attraction can arise between two electrons. To simplify the situation we will start with two electrons and add them to the Fermi sea as proposed by Cooper. The assumptions are:

- The added two electrons interact with each other via an attractive interaction besides having coulomb repulsion. We will assume whatever be the origin and strength of this interaction it overrides the screened coulomb potential.
- The two electrons do not interact with the rest of the electrons in the Fermi sea by any means other than the Pauli Exclusion Principle.

Now to do something with this pair of electrons we need to take a form of wavefunction which would satisfy Bloch's condition and have lowest possible energy i.e they should have equal and opposite momenta.

$$\psi_0(\vec{r_1}, \vec{r_2}) = \sum_k g_k exp(\imath \vec{k} \cdot \vec{r_1} - \imath \vec{k} \cdot \vec{r_2}) \tag{1}$$

where $\vec{k}$ is the momentum of a single electron, the sum runs over all the values of momentum available in the band of interest. g_k is the weight factor corresponding to momentum $\vec{k}$. Now, we will antisymmetrize 1. Anticipating an attractive potential we can safely argue that spatially symmetric wavefunction would have lower energy because symmetric spatial part means electrons are more likely to be close to each other, so have lower energy while an antisymmetric wavefunction would rule out this possibility. Hence we have

$$\psi_0(\vec{r_1}, \vec{r_2}) = \left[\sum_{k > k_F} g_k cos(\vec{k} \cdot (\vec{r_1} - \vec{r_2})) \right] (\alpha(1)\beta(2) - \alpha(2)\beta(1)) \tag{2}$$

Let $r_1 - r_2 = R$ Inserting this into Schrodinger's equation, we get,

$$\sum_{q > k_F} \frac{2\hbar^2 q^2}{2m} \left[g_{\vec{q}} \exp\left(\imath \vec{q} \cdot \vec{R}\right) + g_{\vec{q}} \exp\left(-\imath \vec{q} \cdot \vec{R}\right) \right] +$$

$$\sum_{q > k_F} V(R) g_{\vec{q}} \left[\exp\left(\imath \vec{q} \cdot \vec{R}\right) + \exp\left(-\imath \vec{q} \cdot \vec{R}\right) \right] = E \sum_{q > k_F} \left[g_{\vec{q}} \exp\left(\imath \vec{q} \cdot \vec{R}\right) + g_{\vec{q}} \exp\left(-\imath \vec{q} \cdot \vec{R}\right) \right] \tag{3}$$

where the factor of two before kinetic energy comes because we have two particles.

Now multiply 3 with $\exp\left(-\imath \vec{k} \cdot \vec{R}\right)$ and then integrating with respect to R we land up with

$$(E - 2\epsilon_{\vec{k}}) g_{\vec{k}} = \sum_{q > k_F} V_{\vec{k}\vec{q}} g_{\vec{q}} \tag{4}$$

where, $V_{\vec{k}\vec{q}}$ is given by,

$$V_{\vec{k}\vec{q}} = \frac{1}{\Omega} \int V(\vec{R}) e^{i(\vec{q} - \vec{k}) \cdot \vec{R}} d\vec{R}$$

and Ω is the normalization volume and $\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$.Note, $V_{\vec{k}\vec{q}}$ physically says about the strength of the scattering i.e the amplitude for a momenta pair $(\vec{k}, -\vec{k})$ to get scattered into a momenta pair $(\vec{q}, -\vec{q})$. Now the question boils down to whether we can find a set of $g_{\vec{k}}$ satisfying $E < 2E_F$ and 4. If the answer to the question is affirmative we are actually showing that the Fermi sea is unstable against the formation of at least one bound pair regardless how weak the interaction is provided it is attractive.

To simplify the matter Cooper introduced a cut off such that $V_{\vec{k}\vec{q}} = -V$ upto that cut off value of $\vec{k}$ with constraint that the energy corresponding to this cut off lies $\hbar\omega$ away from E_F, and zero beyond that. The negative sign here is the signature of the phonon mediated attraction. One can of course justify this approximation by some more detailed calculation involving jellium which is beyond the scope of our present treament. Using the above approximation, we can get from 4,

$$g_{\vec{k}} = V\frac{\sum g_{\vec{q}}}{2\epsilon_k - E}$$

Summing over $\vec{k}$, we get after cancelling $\sum g_{\vec{q}}$,

$$\frac{1}{V} = \sum_{k>k_F} \frac{1}{2\epsilon_k - E} \tag{5}$$

This summation can be converted into an integral,

$$\frac{1}{V} = N(0)\int_{E_F}^{E_F+\hbar\omega} \frac{d\epsilon}{2\epsilon_k - E} \tag{6}$$

where $N(0)$ is the density of states of electron with one spin direction at Fermi sea.Note the limits of integration.We assume density of states does not vary over this range. After integrating we get

$$\frac{1}{V} = \frac{N(0)}{2} ln\left(\frac{2E_F - E + 2\hbar\omega}{2E_F - E}\right) \tag{7}$$

In the weak coupling approximation ($N(0)V << 1$ which is to say $2E_F - E << 2\hbar\omega$)which is true for many superconductors we have

$$E \approx 2E_F - 2\hbar\omega \exp\left(-\frac{2}{N(0)V}\right) \tag{8}$$

Behold-the energy is less than $2E_F$ which would be the energy of the two electrons had they not been in interaction with each other. Because of this lowering of energy Fermi sea becomes unstable.This is how an energy gap opens up between the ground state and the excited state. When the thermal energy of the system is not enough to cross the gap, the system will stay in the ground state. This is precisely what happens in a superconducting state and this is why we can achieve such state only below the critical superconducting temperature. The energy gap also explains the existence of supercurrent.If the thermal energy is low electrons can not get scattered by the lattice because they have nowhere to go except to cross the energy gap which is not possible due to low thermal energy.Hence they move in a collective fashion giving rise to supercurrent.And also due to the bosonic feature there's more amplitude to go to the same state rather than an unoccupied state by a factor of $\sqrt{n}$. Hence breaking one pair is unfavourable and to break all pairs you need too much energy!!

3 BCS Ground State

The ground state as described by BCS theory can be written as

$$|\psi_G\rangle = \prod_{\vec{k}} (u_{\vec{k}} + v_{\vec{k}} c^*_{\vec{k}\uparrow} c^*_{-\vec{k}\downarrow})|\phi_0\rangle. \tag{9}$$

where, u_k and v_k are the probability amplitudes that $(\vec{k}\uparrow, -\vec{k}\downarrow)$ will not be occupied by a cooper pair and otherwise respectively. Clearly, they obey

$$|u_k|^2 + |v_k|^2 = 1 \tag{10}$$

as probabilities must add up to one.

Note, the product is over all the available values of k in a band. To illustrate the product, suppose the band has M levels. So, k takes values $k_1, k_2, ..., k_M$. now, if we expand out the product in the above expression then, it looks like,

$$|\psi_G\rangle = (u_{\vec{k_1}}...u_{\vec{k_M}} + v_{\vec{k_1}} c^*_{\vec{k_1}\uparrow} c^*_{-\vec{k_1}\downarrow} u_{\vec{k_2}}...u_{\vec{k_M}} + v_{\vec{k_1}}...v_{\vec{k_M}} c^*_{\vec{k_1}\uparrow} c^*_{-\vec{k_1}\downarrow}...c^*_{\vec{k_M}\uparrow} c^*_{-\vec{k_M}\downarrow})|\phi_0\rangle. \tag{11}$$

From the left the first term acts on vaccum and gives no cooper pair; while the seond term gives one cooper pair because one pair of creation operator $c^*_{\vec{k_1}\uparrow} c^*_{-\vec{k_1}\downarrow}$ acts on vacuum and so on.Finally last term on the rightmost gives M cooper pairs which means all the levels are filled by cooper pairs.So we have a superposition of states of fixed no. of particles i.e $|\psi_G\rangle$ does not represent an eigenstate of Number operator.It represents a state where cooper pairs are popping in and out of the system.

To emphasize the point we recast 9 in following form:

$$|\psi_G\rangle = \sum_{N=1}^{M} \lambda_{2N} |\psi_{2N}\rangle \tag{12}$$

where $|\psi_{2N}\rangle$ is the eigenstate of N cooper pairs.Now, we exploit 10 and rewrite

$$u_{\vec{k}} = |u_{\vec{k}}| \tag{13}$$

$$v_{\vec{k}} = |v_{\vec{k}}| \exp(2\imath\phi) \tag{14}$$

In this fashion we are associating a phase factor of ϕ with the ground state.

Hence, our good old friend 9 will look like,

$$|\psi^{\phi}_{BCS}\rangle = \prod_{\vec{k}} (|u_{\vec{k}}| + |v_{\vec{k}}| \exp(2\imath\phi)\, c^*_{\vec{k}\uparrow} c^*_{-\vec{k}\downarrow})|\phi_0\rangle. \tag{15}$$

Expanding out 15 we could easily show a phase of $2N\phi$ is attached with the eigenstate of 2N particle i.e N cooper pairs.So we can write:

$$|\psi^{\phi}_{BCS}\rangle = \sum_{N=1}^{M} |\lambda_{2N}| \exp(2\imath N\phi)\, |\psi_{2N}\rangle \tag{16}$$

This paves the way to project out the eigenstates of number operator by using following identity:

$$\int_0^{2\pi} d\phi \exp(\imath(N-M)\phi) = 2\pi\delta_{NM} \tag{17}$$

Hence acting on 15 by $\exp(-\imath N\phi)$ and integrating over 0 to 2π angle we have the following:

$$|\psi_{2N}\rangle = \frac{1}{2\pi|\lambda_{2N}|}\int_0^{2\pi} d\phi \exp(-2\imath N\phi)\, |\psi^{\phi}_{BCS}\rangle \tag{18}$$

Note, δ_{NM} picks up the correct state from the R.H.S of 15.

It is noteworthy by integrating over the phase degree of freedom we are making it completely uncertain and this is the cost we have to pay to get a precise value of particle number. On the other hand, keeping phase fixed we have $\Delta N \sim \left(\frac{T_c\langle N\rangle}{T_f}\right)^{0.5} \sim 10^9$. Behold Heisenberg's uncertainty principle is right behind us peeping its head out to enforce

$$\Delta N \Delta\phi \sim 2\pi \tag{19}$$

where $\hbar$ is taken be unity.

Thus we have conjugacy relationship between number of particles and phase. To make our idea concrete let us act by $-\imath\frac{\partial}{\partial\phi}$ on $|\psi^{\phi}_{BCS}\rangle$ and project out the particle eigenstate. We will get

$$\frac{1}{2\pi|\lambda_{2N}|}\int_0^{2\pi} d\phi \exp(-2\imath N\phi)\left(-\imath\frac{\partial}{\partial\phi}\right)|\psi^{\phi}_{BCS}\rangle \tag{20}$$

Doing by parts the first term vanishes while the second term in which derivative acts on exponential yields

$$\frac{-1}{2\pi|\lambda_{2N}|}\int_0^{2\pi} d\phi(-\imath)^2(2N)\exp(-2\imath N\phi)\, |\psi^{\phi}_{BCS}\rangle = 2N|\psi_{2N}\rangle \tag{21}$$

We have following correspondence

$$\hat{N} \leftrightarrow -\imath\frac{\partial}{\partial\phi} \tag{22}$$

In similar fashion, acting $\imath\frac{\partial}{\partial(2N)}$ on 16 we get $\hat{\phi} \leftrightarrow \imath\frac{\partial}{\partial N}$ where N is treated as a continuous variable. The number operator and the phase operator should satisfy the following relations:

$$\imath\dot{\hat{N}} = [\hat{H}, \hat{N}] = \imath\frac{\partial\hat{H}}{\partial\phi} \tag{23}$$

$$\imath\dot{\hat{\phi}} = [\hat{H}, \hat{\phi}] = -\imath\frac{\partial\hat{H}}{\partial N} \tag{24}$$

To conclude this section we remark on the fact that in statistical limit $\frac{\Delta N}{\langle N\rangle}$ goes to 0. So we can neglect the relative fluctuation in N and on the other hand from uncertainty relation we get $\Delta\phi$ to be small, hence in statistical limit we can talk of average N and treat ϕ as a semi classical variable.This is how superconductivity emerges as a macroscopic phenomenon with phase playing a very important role what we will see later in case of Josephson junction and squid.

4 Macroscopic Realisation of Wavefunction

After the triumph of celebrated Schrodinger equation, it was Max Born who first put forward the correct probabilistic interpretation of wavefunction. The modulus squared of the wavefunction according to this prescription yields the probability of finding electron in some region, but once it is found the full charged electron is there-there is no concept of smeared out charged density as we have only one electron. Now suppose some alien guy comes and puts millions of electron in same state (you can say, hey what's about Pauli's exclusion? Hold your breath for a second!!we will come to this point). Then it seems we can employ the classical definition of probability i.e if we probe any region there is a probability of finding n electron out of N if probability is given by n/N.But this works only if N is large, and certainly breaks apart when $N = 1$ which is the usual case for electron. In turn,in case of N being large, we can consider the charge density to be smeared out whose probability distribution follows modulus squared of wavefunction.

Now, recall BCS theory presents us with **Cooper pair**. So in a sense, **Phonon** is our alien guy who binds two electrons of opposite spin and momenta in a pair and the pair behaves as a quasi particle. Clearly, the pair is bosonic in nature. Lo and behold we are not violating the Pauli's exclusion principle and it turns out natureis smart!!.**In fact, the bosonic behaviour is responsible for all these macroscopic realisation of Quantum mechanics. For example we have electromagnetic waves which is macroscopically quite relevant and their quanta photons obey Maxwell equations which is analogous to Schrodinger equation**

With all the arguments given above, we take a leap of faith to come up with following ansatz:

$$\psi(r) = \sqrt{\rho(r)}\exp\left(i\theta(r)\right) \tag{25}$$

where $\rho(r)$ is the charge density.

4.1 Phase becomes Physically Observable

We plug in the ansatz in the expression for probabilty current from the schrodinger equation:

$$\vec{J} = \frac{1}{2m}\left[\psi^*\left(-\imath\hbar\nabla - q\vec{A}\right)\psi + \psi\left(\imath\hbar\nabla - q\vec{A}\right)\psi^*\right] \tag{26}$$

$$\psi^*\nabla\psi = \imath\rho(r)\nabla\theta + \frac{\nabla\rho}{2} \tag{27}$$

$$\frac{-\imath\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = \frac{\hbar}{m}\rho(r)\nabla\theta \tag{28}$$

Hence the expression for current comes out to be

$$\vec{J} = \frac{\hbar}{m}\left(\nabla\theta - \frac{q}{\hbar}\vec{A}\right)\rho \tag{29}$$

We can cast the last equation in a form which illuminates the canonical structure:

$$m\vec{v} = -\imath\hbar\nabla\left(\imath\theta\right) - q\vec{A} \tag{30}$$

where we have used $\vec{J} = \rho\vec{v}$. Now we can identify L.H.S to be kinematic momenta, the term with $\hbar$ to be canonical momenta while $q\vec{A}$ is to identified with momenta of the field it is interacting with.

5 Meissner Effect

In the year 1933 Meissner and Oschenfeld for the first time observed that during the transition to superconducting state magnetic field lines are expelled by the superconductor.This phenomenon of perfect diamagnetism bears the name of Meissner. The following is a schematic diagram showing Meissner effect in a lump (more technically, object having simply connected topology) superconductor:

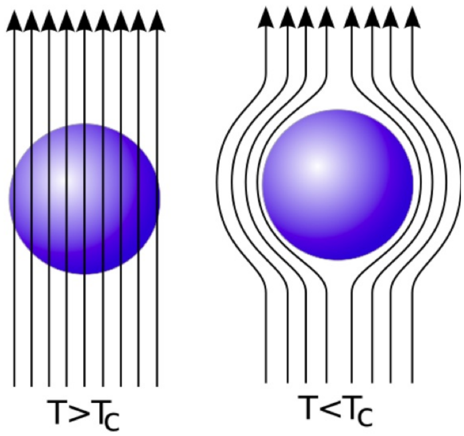

Figure 1: Meissner Effect in Lump Superconductor

Before going into details of Meissner effect we do need to derive a equation which was first given by London brothers on the basis of phenomenology, in the year 1935. And it could quite successfully explain the Meissner effect. To add a historical note, London and London equation was justified later via Landau-Gingburg and BCS theory;initially it was given on purely phenomenological basis. In this report, we would go in a reverse direction and will take a retrospective path. So we are gonna first derive London and London Equation.

5.1 Derivation

We assume to have a time independent situation inside the superconductor.It is justified because charge can neither enter nor get out of the superconductor. Hence we have $\vec{\nabla} \cdot \vec{J} = 0$From 29 we get,

$$\frac{\hbar}{m}\left(\nabla^2\theta - \vec{\nabla}\frac{q}{\hbar}\vec{A}\right)\rho = 0 \tag{31}$$

Note we have assumed charge density to be uniform. We justify it a following way: had not the charge density be uniform we would have terrific electrostatic force inside the superconductor. As a result cooper pairs will be broken up to create normal electrons which in turn move in to neutralise the situation i.e any excess of positive charge. And the process of pair breaking requires energy. So it is energetically highly favourable that a uniform charge density is maintained throughout.

Thus we justify why ∇ does not act on ρ. Now Coulmb Gauge yields (At a first glance we would see A_μ has got a mass, so we can not impose coulomb gauge. But here A_μ is getting mass via spontaneous symmetry breaking-in a level ahead lagrangian still enjoys gauge symmetry!! So there is no problem in choosing coulomb gauge. And here we have $A^0 = 0$-therefore lorentz and coulomb gauge is synonymous in our case!!)

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{32}$$

Plugging this in 31 will yield

$$\nabla^2 \theta = 0 \tag{33}$$

Now we impose the following boundary condition:

- θ is uniform across the boundary.

Invoking **Uniqueness Theorem** we get

$$\theta = constant \tag{34}$$

$$\nabla \theta = 0 \tag{35}$$

Plugging 35 in 29 we get famous London and London equation:

$$\vec{J} = \frac{-\rho q}{m} \vec{A} \tag{36}$$

5.2 Coupling with Maxwell

Maxwell's equation reads in a time independent scenario:

$$\nabla^2 \vec{A} = -\frac{1}{\epsilon_0 c^2} \vec{J} \tag{37}$$

Coupling 37 with 36 we get the ingredient to explain Meissner effect:

$$\nabla^2 \vec{A} = \lambda^2 \vec{A} \tag{38}$$

where $\lambda^2 = \rho \frac{q}{\epsilon_0 m c^2}$.

Lets show the effect of expulsion of magnetic field lines in a spherical superconductor as depicted in the figure at the beginning of the section. The equation is so standard that we can write down its solution just by inspection: (Formally one has to do separation of variable technique to arrive at the solution, here we do not need such sophisticated machinery to arrive at the simple solution)

$$\vec{A} = \vec{A_0} \exp\left(\vec{\lambda} \cdot \vec{r}\right) \tag{39}$$

where we have following constraint:

$$|\vec{\lambda}|^2 = \lambda^2 = \rho \frac{q}{\epsilon_0 m c^2} \tag{40}$$

Hence the expression for magnetic field is:

$$\vec{B} = \nabla \times \vec{A} = \vec{\lambda} \times \vec{A_0} exp\Big(\vec{\lambda}.\vec{r}\Big) \tag{41}$$

Now to note the screening effect take the ratio of the field inside and field outside =exp $(r - R)\,\lambda$ where R is the radius of the sphere.

- Note the field is highest at the surface of the sphere.Clearly, the field actually penetrates the sphere over a radial distance of $\frac{1}{\lambda}$ from the surface.
- λ is called London penetration depth which is of the order of μm.

5.3 Estimation of λ

The electromagnetic radius $2.8 \times 10^{-13} cm$ of the electron is given by

$$mc^2 = \frac{q^2}{4\pi\epsilon_0 r_0} \tag{42}$$

Keeping in mind here q is twice the charge of electron we have

$$\frac{q}{\epsilon_0 mc^2} = \frac{8\pi r_0}{q_e} \tag{43}$$

$$\frac{\rho q}{\epsilon_0 mc^2} = 8\pi N r_0 \tag{44}$$

where we have written $\rho = q_e N$ and N is the number of electrons per cubic cm.

For a metal such as lead having 3×10^{22} electrons per cubic cm $\frac{1}{\lambda}$ would be $2 \times 10^{-5} cm$.

6 Aharanov-Bohm Effect and Flux Quantization

In this section we will revisit the **Meissner** effect but in non simply connected topology.It refers to such a space where not every loop can be continuously shrunk to a point.For example 2D plane with one point removed or toroid in 3D space.The phenemenon of meissner effect becomes quite interesting in such scenario. The first figure depicts the material above critical temperature and

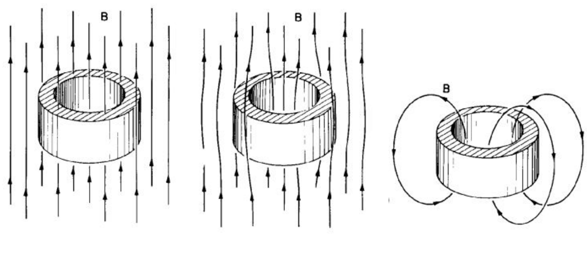

Figure 2: Aharanov-Bohm Effect-Flux Quantisation

a magnetic field pervades the system. As we cool it down below critical temperature to superconducting state we have perfect diamagnetism-field lines are expelled as shown in figure 2.Figure 3 holds the amusement for us!!.Even if we turn off the magnetic field the field lines are still there throgh the hole and it gets kind of traped and that is also in quantized form.

In a superconducting state following holds $\oint \vec{E} \cdot \vec{dl} = 0$, therefore from maxwells equation we can argue

$$\frac{\partial \Phi}{\partial t} = \oint \vec{E} \cdot \vec{dl} = 0 \tag{45}$$

So the flux wont change and field lines will get trapped i.e if you feed magnetic field to such loop in superconducting state then it kind of memorise it so that even when you remove the field, it automatically sets up a supercurrent to keep flux unchanged.

6.1 Why Quantized

The supercurrent we are talking about flows only down to a depth of $\frac{1}{\lambda}$ across the surface.Well inside the body $\vec{J} = 0$.Hence we have from 29

$$\hbar \nabla \theta = q\vec{A} \tag{46}$$

Intergrating over a closed loop R.H.S yields flux Φ through the loop by virtue of standard stoke's theorem.

Hence we have

$$\oint \vec{\nabla}\theta \cdot \vec{ds} = \frac{q}{\hbar}\Phi \neq 0 \tag{47}$$

This is where non simply connected topolgy comes into picture. In a simply connected topology L.H.S is always zero as curl of a gradient is zero in such topology. But here we can not demand single valuedness of phase, rather what we can physically demand is wave function is single valued!! In fact wavefunction is allowed to pick up a phase factor of $2n\pi$ where $n \in \mathbb{Z}$. [**This is actually the Berry phase**]. So we can write the L.H.S of 47 as following:

$$\oint \vec{\nabla}\theta \cdot \vec{ds} = 2n\pi \tag{48}$$

Plugging this into 47 yields

$$\Phi = \frac{2n\pi\hbar}{q} = n\Phi_0 \tag{49}$$

Note the flux is indeed quantized where the quanta Φ_0 is given by:

$$\Phi_0 = \frac{2\pi\hbar}{q} = \frac{\pi\hbar}{q_e} = 2 \times 10^{-7} gauss - cm^2 \tag{50}$$

where q is charge of the cooper pair i.e twice of electronic charge.

To visualise such a flux, consider a tiny cylinder, a tenth of a mm in diameter; the magnetic field inside it when it contains this amount of flux is about 1% of earth's magnetic field.

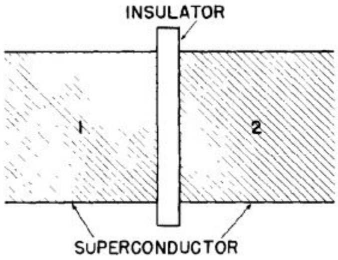

Figure 3: Josephson Junction

7 Josephson Junction

One of the amazing applications of superconductivity is Josephson junction. Consider the above set up in figure 3: two superconducting regions are joined by a thin layer of insulator.

We join the other ends of the superconductor to a battery or short them and the question we will address whether there flows a finite amount of current across the junction.Because of insulator one would expect to see no current but if the layer is thin enough (say $10^{-7}m$) i,e comparable to the Fermi wavelength of electron strictly speaking cooper pair then there is a finite probability that cooper pair can tunnel through from region 1 to region 2 to give experimentally detectable amount of current (of the order of nA)

7.1 Governing Equation

If we create voltage difference V across the junction then the following two Schrodinger like equation will govern the system:

$$\imath\hbar\frac{\partial\psi_1}{\partial t} = \frac{qV}{2}\psi_1 + K\psi_2 \tag{51}$$

$$\imath\hbar\frac{\partial\psi_2}{\partial t} = -\frac{qV}{2}\psi_2 + K\psi_1 \tag{52}$$

The key points to note that

- This is a kind of two level system with K term corresponding to the hopping/ flip-flop probability of cooper pair to hop from region1 to 2 and vice-versa-the offdiagonal elements of hamiltonian.K is the characteristic of the junction.
- It mimics the Schrodinger equation yet the momentum part is missing. And this is due to the fact charge density is uniform resulting to the fact $\nabla\psi$ can be taken to be 0.
- This ψ has an uncanny similarity with the order parameter defined in Landau-Gingburg theory.There $\nabla\psi$ is 0 to minimise free energy of the system.

Now plug in the following ansatz in 51:

$$\psi_1 = \sqrt{\rho_1} \exp(\imath\theta_1) \tag{53}$$

$$\psi_2 = \sqrt{\rho_2} \exp(\imath\theta_2) \tag{54}$$

with $(\theta_2 - \theta_1) = \delta$

The 51 yields:

$$\imath\hbar \frac{1}{2\sqrt{\rho_1}} \exp(\imath\theta_1) \frac{\partial \rho_1}{\partial t} - \hbar\sqrt{\rho_1} \exp(\imath\theta_1) \frac{\partial \theta_1}{\partial t} =$$

$$\frac{qV}{2}\sqrt{\rho_1} \exp(\imath\theta_1) + K\sqrt{\rho_2} \exp(\imath\theta_2) \tag{55}$$

Now we will use $(\theta_2 - \theta_1) = \delta$ and separate real and imaginary part to arrive at

$$\frac{\partial \rho_1}{\partial t} = \frac{2K}{\hbar}\sqrt{\rho_1\rho_2} \sin(\delta) \tag{56}$$

$$\frac{\partial \theta_1}{\partial t} = -\frac{K}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}} \cos(\delta) - \frac{qV}{2\hbar} \tag{57}$$

where the first one comes by equating imaginary part while the second one is from real part.

Similarly 52 yields

$$\frac{\partial \rho_2}{\partial t} = -\frac{2K}{\hbar}\sqrt{\rho_1\rho_2} \sin(\delta) \tag{58}$$

$$\frac{\partial \theta_2}{\partial t} = -\frac{K}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}} \cos(\delta) + \frac{qV}{2\hbar} \tag{59}$$

7.2 Evolution of Charge Density

The 56 and 58 say $\frac{\partial \rho_1}{\partial t} = -\frac{\partial \rho_1}{\partial t}$. But,wait,shouldn't they be zero with ρ_i fixed at some ρ_0.The subtlety lies here that the above equations tell us only how the charge densities would start to change before the circuit is complete i.e before the cooper pair leaving region 2 has walked along the external wire and got into region 1.If we consider this, then there is a finite current flowing across the junction given by

$$J = \frac{2K}{\hbar}\sqrt{\rho_1\rho_2} \sin(\delta) = J_0 \sin(\delta) \tag{60}$$

Now note such a current can never charge up region 2 or discharge any region 1 due to the simple fact we have closed circuit,hence whatever leaves must come in-it can not get lost in mid way.In short, the picture is

- $\frac{\partial \rho_1}{\partial t} = -\frac{\partial \rho_1}{\partial t} = 0$
- Still we have a current across the junction given by $J_0 \sin(\delta)$.

7.3 Evolution of Phase

Substracting the 57 from 59 yields:

$$\frac{\partial \delta}{\partial t} = \frac{d\delta}{dt} = \frac{qV}{\hbar} \tag{61}$$

where partial and total derivative is same as δ does not have space dependence which stems from the fact $\nabla \theta_i = 0$ for a lump superconductor.

Integrating 61 with respect to time we get

$$\delta(t) = \delta(0) + \frac{q}{\hbar} \int dt V(t) \tag{62}$$

Now in the next two subsections we are going to consider two choices for the potential function $V(t)$.

7.4 DC Josephson Effect

The simplest possible choice is DC voltage i.e $V(t) = V_0$.

Using 62 what we get is

$$J = J_0 \sin\left(\delta(0) + \frac{qV_0 t}{\hbar}\right) \tag{63}$$

Now the important point is since $\hbar$ is a small number compared to ordinary voltage and time, the net current oscillates very rapidly and we would not have any current in experimental time scale.(except a small current due to thermal fluctuation and conduction by normal electron not cooper pair).To get a detectable amount of current we need to make the time dependence vanish. Hence we need $V_0 = 0$.Thus with no DC voltage there is a current $J_0 \sin(\delta)$ which can take any value between J_0 and $-J_0$.But if you try to put up a voltage you wont get any current.This startling behaviour has been observed experimentally.[1]

7.5 AC Josephson

Now we will couple a very small AC potential to a large DC potential.So we have

$$V(t) = V_0 + v \sin(\omega t) \tag{64}$$

where $v << V_0$. Integrating we have

$$\delta(t) = \delta(0) + \frac{qV_0 t}{\hbar} + \frac{q}{\hbar}\frac{v}{\omega} \sin(\omega t) \tag{65}$$

The expression for the current becomes

$$J = J_0 \sin\left(\delta(0) + \frac{qV_0 t}{\hbar} + \frac{q}{\hbar}\frac{v}{\omega} \sin(\omega t)\right) \tag{66}$$

Now the third term of the argument of sine is indeed small, so we can make use of the following approximation

$$\sin(x + dx) = \sin x + dx \cos x \tag{67}$$

[1] P.W.Anderson and J.M.Rowell,*Phys. Rev. Letters* **10**,230 (1963)

Using 67, 66 can be written as following

$$J = J_0 \left[\sin\left(\delta(0) + \frac{qV_0 t}{\hbar}\right) + \frac{q}{\hbar}\frac{v}{\omega}\sin(\omega t)\cos\left(\delta(0) + \frac{qV_0 t}{\hbar}\right)\right] \tag{68}$$

Using trigonometric identity we can recast 68 in a form which can be directly interpreted as per as experimental consideration is concerned:

$$J = J_0 \left[\sin\left(\delta(0) + \frac{qV_0 t}{\hbar}\right) + \frac{qv}{2\hbar\omega}\left(\sin\left(\omega t + \frac{qV_0 t}{\hbar} + \delta(0)\right) - \sin\left(\frac{qV_0 t}{\hbar} + \delta(0) - \omega t\right)\right)\right] \tag{69}$$

Note the first two term vanishes due to rapid oscillation due to the fact $\hbar$ is a small number compared to timescale and potential scale of experiment. Now if we carefully focus on the third term we see it becomes time independent at $\omega = \frac{qV_0}{\hbar}$. So in the current frequency graph we have a sharp resonance behaviour at $\omega = \frac{qV_0}{\hbar}$.Otherwise the current would drop to zero!!This has also been experimentally verified.[2]

8 Josephson Effect in presence of Magnetic Field

In presence of magnetic field, vector potential get coupled with momentum operator via Pierls substitution and the wavefunction ψ picks up a berry phase. Hence we have (I am not going into the details of the proof as it has been covered in QM class itself)

$$\psi = \psi_0 \exp\left(\imath \frac{q}{\hbar}\int_P^Q \vec{A}\cdot\vec{ds}\right) \tag{70}$$

where ψ_0 is twavefunction in absence of A_μ.

In absence of magnetic field,across the junction the phase continuously varies from θ_1 to θ_2.But with magnetic field present, the junction can support additional Berry phage given by the line integral of $\vec{A}$ across the junction.Hence effectively we have

$$\theta_2 = \theta_1 + \delta' - \frac{2q_e}{\hbar}\int_P^Q \vec{A}\cdot\vec{ds} \tag{71}$$

where PQ is the insulating link and note copper pair has $-2q_e$ charge.

$$\delta' = (\theta_2 - \theta_1) + \frac{2q_e}{\hbar}\int_P^Q \vec{A}\cdot\vec{ds} = \delta + \frac{2q_e}{\hbar}\int_P^Q \vec{A}\cdot\vec{ds} \tag{72}$$

Hence the correct expression for the current would be

$$J = J_0 \sin(\delta') = J_0 \sin\left(\delta + \int \frac{2q_e}{\hbar}\vec{A}\cdot\vec{ds}\right) \tag{73}$$

where the integral is taken across the junction.Note that the current is now manifestly gauge invariant which is not so obvious in 60.Also note, we get 60 from 73 by putting $\vec{A} = 0$.

[2]S.Shapiro,*Phys. Rev. Letters*,**80** (1963)

9 SQUID

Squid stands for Superconducting Quantum Interference Device.We are all familiar with the quantum interference between probability amplitude.Here we go one step ahead and talk about quantum interference of current and this is possible again due to macroscopic nature of the system and bosonic ground state for which we can treat probability amplitude as square root of charge density.

9.1 DC SQUID

Consider the following scenario where we have two junctions in parallel:

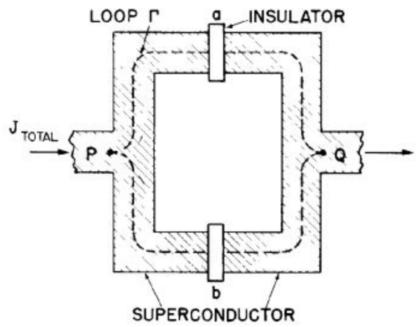

Figure 4: DC SQUID

Clearly two currents pick up following phases:

$$\Delta\phi_{upper} = \delta_a + \frac{2q_e}{\hbar}\int_{upper} \vec{A}\cdot\vec{ds} \tag{74}$$

$$\Delta\phi_{lower} = \delta_b + \frac{2q_e}{\hbar}\int_{lower} \vec{A}\cdot\vec{ds} \tag{75}$$

Now use the fact inside superconductor $\vec{A} = 0$, therefore we can extend the integrals across the link to integrals from point P to Q.

Now $\Delta\phi_{lower} = \Delta\phi_{upper}$ yields:

$$\delta_b - \delta_a = \frac{2q_e}{\hbar}\oint \vec{A}\cdot\vec{ds} = \frac{2q_e}{\hbar}\Phi \tag{76}$$

where Φ is the magnetic flux through the loop.

To keep the maths simple we write

$$\delta_a = \delta_0 + \frac{q_e\Phi}{\hbar} \tag{77}$$

$$\delta_b = \delta_0 - \frac{q_e\Phi}{\hbar} \tag{78}$$

Hence the expression for current is:

$$J = J_0\left[\sin(\delta_a) + \sin(\delta_b)\right] \tag{79}$$

$$J = 2J_0 \sin(\delta_0) \cos\left(\frac{q_e \Phi}{\hbar}\right) \tag{80}$$

The expression is holding surprise for us. We see a beat phenomenon right in front of us.The following graph will elucidate the point:

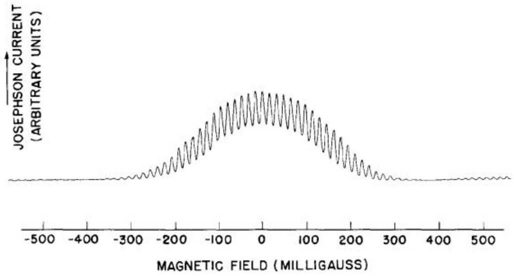

Figure 5: Current vs Flux Behaviour-BEATS

The rapid oscillation in the graph is due to cosine term while it gets modulated by sine term because $\frac{q_e \Phi}{\hbar} >> \delta_0$ and we indeed observe beats beacuse the interference emerges owing to superposition of currents with frequencies very near to each other.

The maximum current for any given Φ is given by

$$J_{max} = 2J_0 |\cos\left(\frac{q_e \Phi}{\hbar}\right)| \tag{81}$$

The maximum current will change with Φ will have maxima at $\Phi = n\pi\frac{\hbar}{q_e} = n\Phi_0$.**It is quite interesting that current will be maximum just when the flux linkage has precisely quantized value!!**

Magnetometer We can build sensitive magnetometer using SQUID. Note the difference between two consecutive maxima in current vs flux graph is just the flux quanta which is $O(10^{-7} gauss - cm^2)$.So if we can make a pair of junction enclosing an area of say $1mm^2$, and if we are certain to resolve $\frac{1}{10}$ th of the way between two peaks then we can measure field upto the order of 10^{-10} gauss which is amazingly a very tiny number.In fact we can increase the number of parallel junctions to achieve higher sensitivity just like having more than 2 slits in interference experiment leads to sharp contrast between dark and bright spots.As an aside it can be mentioned that our braian and many biological system has magnetic field of the order of 10^{-10} gauss,so SQUID comes into rescue when we try to measure those fields with such precision.

9.2 Rf Squid

This is another genre of SQUID operated in radio frequency regime.Following is the circuit diagram:

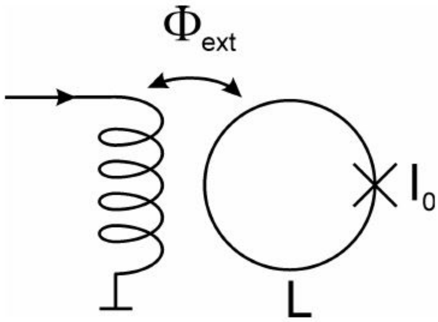

Figure 6: Rf SQUID-Schematic Circuit

The primary coil induces a magnetic fiux (Φ_{ext} through the superconducting loop.A supercurrent starts to flow in the loop to oppose the flux through the loop according to lenz's law. From 73 we can argue the junction can support a phase difference $\delta' = 2\pi\frac{\Phi}{\Phi_0}$ where Φ is the net flux through the loop.(Here we have $\theta_2 = \theta_1$ as we have same material on both sides of the junction)

If the self inductance of the loop is L then current flowing through the loop is $\frac{(\Phi-\Phi_{ext})}{L}$.Due to Kirchhoff's law this should equal to the current across the junction i.e Josephson current which is given by

$$I = I_0 \sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \tag{82}$$

where the direction of current will be such that the flux it produces always opposes the flux due to primary coil.

The following graph depicts the amount of current it can support: Note in

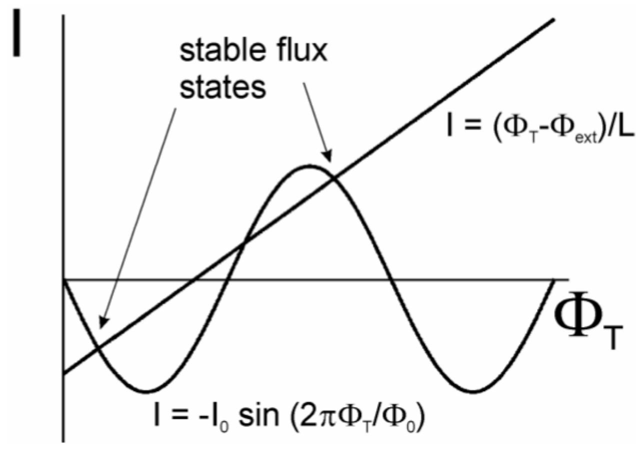

Figure 7: Current vs Net Flux-Stability

graph we have slight notational change-$\Phi_T = \Phi$, net flux.And the sign of the current is to denote opposing behaviour.

We see for a given Φ_{ext} it can not support arbitrary amount of current;It can support some specific values corresponding to the intersection points of the graph.But there is a catch in the story.If we think beyond classically then interesting behaviour can pop up!!To treat the scenario quantum mechanically we promote flux to a quantum variable and try to wite down potential as a function of flux.

The energy associated with Josephson current is:

$$\int I(t)V(t)dt = \frac{\hbar}{2q_e}\int I(t)\frac{d\delta}{dt}dt = \frac{\hbar}{2q_e}\int I_0 \sin(\delta)\, d\delta = -E_0\cos(\delta) \tag{83}$$

where we have used 61 and $E_0 = \frac{\hbar I_0}{2q_e}$

The energy stored in the magnetic field is $\frac{1}{2L}\left(\Phi - \Phi_{ext}\right)^2$, this is the harmonic term of the potential. As a whole we can write,

$$U(\Phi) = -E_0\cos\left(2\pi\frac{\Phi}{\Phi_0}\right) + \frac{1}{2L}\left(\Phi - \Phi_{ext}\right)^2 \tag{84}$$

Let q be the conjugate variable of phase which in fact relates to charge.The junction capacitance behaves as mass of the system. Hence we can write down the Hamiltonian of the system:

$$\hat{H} = \frac{\hat{q}^2}{2m} - E_0\cos\left(2\pi\frac{\hat{\Phi}}{\Phi_0}\right) + \frac{1}{2L}\left(\hat{\Phi} - \Phi_{ext}\right)^2 \tag{85}$$

Now note if we take $\frac{\Phi_{ext}}{\Phi_0} = n + \frac{1}{2}$ then we have

$$\cos\left(2\pi\frac{\Phi_{ext} - \Phi}{\Phi_0}\right) = \cos\left(2\pi\frac{\Phi + \Phi_{ext}}{\Phi_0}\right) \tag{86}$$

where we have used the identity:$\cos\left[(2n+1)\pi - \theta\right] = \cos\left[(2n+1)\pi + \theta\right]$.

Hence, we have

$$U\left(\Phi_{ext} - \Phi\right) = U\left(\Phi_{ext} + \Phi\right) \tag{87}$$

Lo and behold-around the $\Phi = \Phi_{ext}$ we have symmetric double well potential:

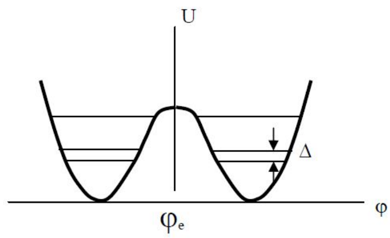

Figure 8: Symmetric Double Well Potential

Qubit-Quantum Computation The two lowest eigenstates of the Hamiltonian 85 $|0\rangle$ and $|1\rangle$ can be chosen to be qubits where the controlling parameter is external flux Φ_{ext}. Classically the system would stabilise itself in one of the minima of the potential which are degenerate. This will also happen if the barrier is too high to effect any tunnelling in quantum scenario. But if there is tunnelling, we would expect the degeneracy to be lifted and there would be splitting we would have following states:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{88}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \tag{89}$$

with the second state being higher in energy. Quantum computer takes advantage of this tunnelling feature which is beyond the scope of this project. It is noteworthy that superconductor being an emblem of QM operating on a macroscopic scale helps to implement quantum computation in real life practice.

10 Dynamics of Superconductivity

For the sake of completeness this section has been put up.It describes the equation that governs the dynamics of superconductivity. In fact we can show it behaves same as a charged fluid under the influence of electric and magnetic field.

Let us start with the Schrodinger equation and separate out its imaginary and real part after plugging in our old ansatz $\psi = \sqrt{\rho}\exp(\imath\theta)$ in the following:

$$\imath\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-\imath\hbar\nabla - q\vec{A}\right)^2\psi + q\phi\psi \tag{90}$$

The imaginary part of 90 would look like following:

$$\hbar\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t}\exp(\imath\theta) =$$

$$\left(\frac{1}{2m}(-\imath\hbar)^2\left(\sqrt{\rho}\nabla^2\theta + 2\nabla\theta\frac{1}{2\sqrt{\rho}}\nabla\rho\right) + \frac{1}{2m}\left(2\hbar q\vec{A}\frac{1}{\sqrt{\rho}}\nabla\rho\right)\right)\exp(\imath\theta)$$

$$-\frac{\hbar}{2m}\sqrt{\rho}\exp(\imath\theta)\nabla q\vec{A} \tag{91}$$

Rearranging the terms we get,

$$\frac{\partial\rho}{\partial t} = -\rho\vec{\nabla}\cdot\left(\frac{\hbar}{m}\nabla\theta - \frac{q}{m}\vec{A}\right) - \left(\frac{\hbar}{m}\nabla\theta - \frac{q}{m}\vec{A}\right)\nabla\rho \tag{92}$$

Upon simplification it would yield the famous continuity equation

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{v}) \tag{93}$$

On the other hand, the real part of 90 would be

$$\imath\hbar\rho(r)\imath\frac{\partial\theta}{\partial t}\exp(\imath\theta) =$$

$$\frac{1}{2m}(-\imath\hbar)^2\left[\imath^2(\nabla\theta)^2\sqrt{\rho} + \nabla^2\sqrt{\rho}\right]\exp(\imath\theta)$$

$$+ q^2\vec{A}^2\sqrt{\rho}\exp(\imath\theta) + \frac{1}{2m}\left[2\imath\hbar q\vec{A}\cdot\sqrt{\rho}\,(\imath)\,\nabla\theta\exp(\imath\theta)\right] + q\phi \tag{94}$$

where we have used

$$\imath\left(q\vec{A}\nabla\psi - \nabla(q\vec{A})\psi\right) = 2\imath q\vec{A}.\nabla\psi - \imath\psi\nabla q\vec{A} \tag{95}$$

with the fact second term is always imaginary!!

Upon simplification i.e dividing by $\sqrt{\rho}\exp(\imath\theta)$, we will get

$$\hbar\frac{\partial\theta}{\partial t} = -\frac{m}{2}v^2 + \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\nabla^2\left(\sqrt{\rho}\right) - q\phi \tag{96}$$

where

$$m\vec{v} = \hbar\nabla\theta - q\vec{A} \tag{97}$$

and we do the following trick to get the $m\frac{v^2}{2}$ term:

$$\frac{1}{2m}\left[\hbar^2\left(\nabla\theta\right)^2 - 2\hbar q\vec{A}\nabla\theta + q^2\vec{A}^2\right] = \frac{1}{2m}\left(\hbar\nabla\theta - q\vec{A}\right)^2 = v^2 \tag{98}$$

Now lets shade light on 96

- It is is energy conservation if we identify $\hbar\theta$ as velocity potential keeping the hindsight of hydrodynamics equation of motion.
- The last term contributes only at the junction otherwise it can be neglected as in a lump of superconductor ρ can safely be assumed to be uniform.

Now we will recast this equation in a more familiar form where it will manifestly behave as a charged fluid:

Take the gradient of 96, express $\nabla\theta$ using 97 and use the following vector identity:

$$\frac{1}{2}\nabla\left(\vec{v}\cdot\vec{v}\right) = \vec{v}\times\left(\nabla\times\vec{v}\right) + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} \tag{99}$$

we get

$$\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}.\nabla\right)\vec{v} = \frac{q}{m}\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right) - \vec{v}\times\left(\nabla\times\vec{v}\right) + \nabla\frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho}}\nabla^2\left(\sqrt{\rho}\right) \tag{100}$$

Note

$$\nabla\times\vec{v} = -\frac{q}{m}\nabla\times\vec{A} = -\frac{q}{m}\vec{B} \tag{101}$$

$$\frac{dv}{dt}|_{comoving} = \frac{\partial\vec{v}}{\partial t} + \left(\vec{v}.\nabla\right)\vec{v} \tag{102}$$

Hence we have following:

$$m\frac{dv}{dt}|_{comoving} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) + \nabla\frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho}}\nabla^2\left(\sqrt{\rho}\right) \tag{103}$$

This equation is nothing but newton's law for a charged fluid in electromagnetic fluid.The first term represents the lorentz force while the second term is quantum mechanical in origin.The other equation 101 tells us the fluid behaves ideally.

There is always a sanity check in this equation101. as curl of velocity has zero divergence we have $\nabla \cdot \vec{B} = 0$ which is again one of the Maxwell's equations.

There is a catch in the whole story. As we know electrostatic force always tends to maintain the charge density to be uniform in superconducting state there should be another term proportional to $\nabla (\rho - \rho_0)^2$.This term did not appear in our treatment because we are dealing in the realm of independent particle approximation.

11 What Lie Ahead

The age of superconductivity is dawning;holding promises for us-to say few:

- Materials like ceramics, show superconductivity at considerably at higher temperatures than that predicted by BCS theory. One typical example of $T_c \sim 150K$.Clearly BCS theory does not work for it, and people are trying come up with a solution.
- Superconducting wires are a great option for transmitting power without any significant loss, however, flexibility of the material, value of T_c, ability to handle large current density and above all, cost are troublesome issues.

12 Reference

- Feynman Lectures in Physics Vol.3
- Michael Tinkham -Introduction to superconductivity.
- Phillips Advanced Solid State Physics.
- P.W.Anderson and J.M.Powell, *Phys. Rev. Letters* **10**,230 (1963)
- S.Shapiro,*Phys. Rev. Letters*11,80(1963)
- Cooper Phys. Rev. **104**,1189 (1956)
- Bardeen Cooper Schreifer, Phys. Rev. **108**,1175 (1957)
- G.P.Berman, A.R.Bishop, A.A.Chumak, D.Kinion, V.I.Tsifrinovich; Measurement of the Josephson Junction Phase Qubits by a Microstrip Resonator-http://arxiv.org/abs/0912.3791

13 Acknowledgement

The author acknowledges a debt of gratitude to Dr. S.Lal for his beautiful and lucid illustration of subtleties involved in Superconductivity. The author has really enjoyed doing the project which would not be possible without the help and encouragement he got from his instructor Dr. Lal.